Identification of Time and Risk Preferences in Buy Price Auctions

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Abstract

Buy price auctions merge a posted price option with a standard bidding mechanisms, and have been used by various online auction sites including eBay and GMAC. A buyer in a buy price auction can accept the buy price to win with certainty and end the auction early. Intuitively, the buy price option may appeal to bidders who are risk averse or impatient to obtain the good, and a number of authors have examined how such mechanisms can increase the seller's expected revenue over standard auctions. We show that data from buy price auctions can be used to identify bidders' risk aversion and time preferences. We develop a private value model of bidder behavior in a buy price auction with a temporary buy price. Bidders arrive stochastically over time, and the auction proceeds as a second-price sealed bid auction after the buy price disappears. Upon arrival, a bidder in our model is allowed to act immediately (i.e. accept the buy price if it is still available, or place a bid) or wait and act later. Allowing for general forms of risk aversion and impatience, we first characterize equilibria in cutoff strategies and describe conditions under which all symmetric pure-strategy subgame-perfect Bayesian Nash equilibria are in cutoff strategies. Given sufficient exogenous variation in auction characteristics such as reserve and buy prices and in auction lengths, we then show that the arrival rate, valuation distribution, utility function, and time-discounting function in our model are all nonparametrically identified. We also develop extensions of the identification results for cases where the variation in auction characteristics is more limited.

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1 Introduction

This paper studies identification of bidder preferences in single unit buy price (BP) auctions. BP auctions merge a posted price selling environment with an auction environment, and have been used by eBay (in their "Buy-it-Now" auctions), GMAC, and other organizations as an alternative to standard first or second price auctions. We show that data from BP auctions can be particularly informative about risk aversion and time preferences among potential bidders, in a way that standard auctions are not. As a result, it is possible to recover bidder preferences from widely available observational data, or carry out experiments to obtain appropriate data and recover those preferences. Knowledge of these preferences may be important for various reasons, perhaps foremost to assess optimal auction design.

There is a large theoretical literature that shows how BP auctions can increase expected revenue over standard auctions; see Budish and Takeyama (2001), Mathews (2004), Mathews and Katzman (2006), Hidvégi, Wang and Whinston (2006), Gallien and Gupta (2007), Wang, Montgomery, and Srinivasan (2008), and Reynolds and Wooders (2009). BP auctions allow some or all potential bidders to purchase the item immediately at a posted buy price. If this does not happen, a standard auction is held. Intuitively, the buy price option may appeal to bidders who are risk averse or impatient to obtain the good. The existing theoretical models typically assume risk aversion, impatience, or both, on the part of bidders, and show that BP mechanisms can increase expected revenue to the seller. Since bidders' decisions in BP auctions depend on their risk aversion and their impatience, one might conjecture that observed data from BP auctions could be informative about risk aversion and impatience.

Many of the existing theoretical models of BP auctions abstract significantly from the specific mechanisms used in practice. For example, some of the models are purely static, whereas in practice many BP auctions have two phases, a buy price phase and a bidding phase, with particular rules about when the bidding phase starts and how long it lasts. The models of Mathews (2004) and Gallien and Gupta (2007) do feature sequential arrival of bidders and an auction format closely modeled on eBay auctions, but impose specific parametric forms on time discounting or risk-aversion. We first develop a theoretical model for a BP auction which captures some key dynamic features of real-world BP auctions and allows for general forms of both risk aversion and impatience, but leads to a tractable equilibrium and relatively straightforward identification results. In our model, bidders have independent private values and arrive according to a time-varying Poisson process. Any potential bidder who arrives in the buy price phase can purchase the good at the buy price (thereby winning the item and ending the auction), can bid (thereby initiating the bidding phase), or can wait. Potential bidders who arrive during the bidding phase (or bidders who arrived during the buy price phase and have waited) can place bids. The bidding phase lasts for a fixed amount of time and is modeled as a second-price sealed bid auction.[1] We describe conditions under

[1]Our fixed length bidding phase differs from the setup of Mathews (2004) and Gallien and Gupta (2007), who

which all symmetric, pure-strategy, subgame-perfect Bayesian Nash equilibria (BNE) of this game are in cutoff strategies, where a potential bidder arriving in the buy price phase accepts the posted price if her valuation is sufficiently high.

Having characterized equilibrium strategies for potential bidders in the auction, we consider identification. The key idea is that bidders facing a choice between buying the good at the buy price or bidding for the good are choosing between a certain prospect and a lottery. If there is sufficient variation in these lotteries and we can determine their certainty equivalents, we can recover bidders' preferences. In our model, bidders are heterogeneous in their valuations, but have common utility and time-discounting functions. This allows bidders to be risk averse, impatient, or both. Our setup imposes some restrictions on the nature of bidder preferences, but under these assumptions, we show that the arrival rate function, the distribution of valuations, the utility function, and the time-discounting function are nonparametrically identified under an assumption of exogenous variation in the auction setup (e.g. reserve and buy prices), and some support conditions. The assumption that reserve and buy prices vary exogeneously is somewhat strong, but provides a natural starting point for identification analysis and could be relaxed in various ways. Our results could also be used by sellers (or economists) who wish to experiment with reserve and buy prices in order to learn about the preferences of buyers. Elfenbein, Fisman, and McManus (2012) and Einav, Kuchler, Levin, and Sundaresan (2015) provide evidence that sellers on eBay may experimentally vary auction features, presumably to guide their auction design. We also show that the model is overidentified, in the sense that it imposes testable restrictions on the distribution of observed data. In addition, if the support conditions are not fully met, then certain local versions of the structural objects are identified.

Although our model captures many features of real-world BP auctions, it does differ in some details from the BP auctions used by both eBay and those used by GMAC. We show that under some additional assumptions, primarily used to guarantee that bidders use cutoff strategies immediately upon arrival, our identification arguments can be extended to these two cases. The extended identification results for eBay-style Buy-it-Now auctions are being used in ongoing empirical work (Ackerberg, Hirano, and Shahriar, 2006). As in many other papers on identification in auctions, we restrict attention to cases where auctions are isolated (ruling out multiple sequential auctions, for example). This likely limits the applicability of our results to many eBay markets, but there are some cases where this assumption may be plausible, and our work provides a starting point for possible future extensions to sequential auctions models in the spirit of Jofre-Bonet and Pesendorfer (2003) or Backus and Lewis (2009).

Our findings contribute to the literature on identification of auction models, and more generally to the literature on recovering risk aversion and other features of preferences from revealed be-

assume a fixed overall length of the auction (similar to eBay). The reason we consider this alternative is because it makes our basic identification results most straigtforward. In Section 5, we extend our identification results to eBay-style models.

havior. Beginning with Guerre, Perrigne, and Vuong (2000), Li, Perrigne, and Vuong (2002), and Athey and Haile (2002), a large literature has emerged exploring identification in various auction formats.[2] If bidders are risk averse, identification becomes much more challenging in these formats; see, e.g. Campo, Guerre, Perrigne, and Vuong (2011), Bajari and Hortacsu (2005), Campo (2012), Perrigne and Vuong (2007), Lu and Perrigne (2008), Athey and Haile (2007), and Guerre, Perrigne, and Vuong (2009). To our knowledge, our paper is the first to examine identification of bidder preferences in BP auctions, and our identification results indicate that these auctions can provide considerable information on bidder risk preferences. Moreover, our results show that bidder propensities to accept buy prices can be used to infer *both* their risk aversion and time preferences. The importance of handling these jointly has been emphasized by Andersen, Harrison, Lau, and Rutström (2008) in a parametric setting. If there is a temporal component to a risky option, ignoring time preferences can bias estimates of risk aversion.

Chiappori, Gandhi, Salanie, and Salanie (2009) provides a general framework for studying nonparametric identification of risk preferences. Our approach is similar in spirit but we also incorporate time preferences and seek to jointly identify risk and time preferences nonparametrically. As in Chiappori, Gandhi, Salanie, and Salanie (2009) and other work on nonparametric identification of structural models (for a recent survey see Matzkin (2013)), an important restriction of our model is that unobserved heterogeneity across bidders is summarized by a scalar. In our setting, this scalar unobserved heterogeneity takes the form of variation across bidders in their valuation of the item (following the majority of the auction literature), while the utility and time-preference functions are assumed to be identical across bidders. It may be possible to consider alternative models where the scalar unobserved heterogeneity enters other parts of the model, such as heterogeneity in risk preferences. However, it would likely be very challenging to allow multi-dimensional unobserved heterogeneity unless one imposes other restrictions on the model.

2 Model and Equilibrium

We start with a simple continuous time, independent private value (IPV) BP auction.[3] The auction starts at time 0, has a reserve price (minimum bid) $r \in [0, \infty)$, and a buy price $p \in [r, \infty)$. At times $t > 0$, potential bidders arrive at the auction according to a Poisson process with rate $\lambda(t)$. These potential bidders have private valuations v drawn independently from the distribution $F_V(v)$.

There are two phases in the auction. At any time t in the the first phase (the buy price phase), any potential bidder who has previously arrived at the auction can take one of the following actions:

1. Immediately purchase the object at p (Accept the BP). In this case, the auction ends.

[2] See Haile and Tamer (2003), Bajari and Hortacsu (2003), Song (2004), Adams (2004, 2007, 2009), Gonzales, Hasker, and Sickles (2004), Canals-Cerda and Pearcy (2013), Nekipelov (2008), Zeithammer and Adams (2010), and Backus and Lewis (2009), among others. Athey and Haile (2007) survey identification results for auction models.

[3] See Shahriar (2008) for a model of BP auctions with common values.

2. Submit a sealed bid $b > r$ for the object (Reject the BP). In this case, the buy price phase ends, and the auction immediately enters the second phase (the bidding phase).

The bidding phase lasts for fixed length $\tau > 0$. During the bidding phase, potential bidders no longer have the option to purchase the object immediately at p. Other potential bidders who either have already arrived, or who arrive during the bidding phase, can also submit a sealed bid $b > r$ for the object. These sealed bids can be placed at any time during the bidding phase.

At the end of the bidding phase, the auction ends and the object is awarded to the bidder who has placed the highest sealed bid. The winning price is the maximum of either the reserve price r or the highest sealed bid of the other bidders. We assume that bidders do not directly observe the actions or arrivals of other bidders. However, we assume that any bidder who is present at the auction at t knows whether the auction is currently in the buy price phase or the bidding phase, and if the latter, that the bidder knows at what point in time the auction entered the bidding phase.

In the terminology of Gallien and Gupta (2007), our auction features a "temporary buyout option," which disappears once the buy price is rejected and the bidding phase begins.[4] However, as long as no potential bidder accepts or rejects the BP, the auction continues indefinitely. There are a number of possible variations on this mechanism. For example, we could consider a design where if by time $T - \tau$, no bidder has accepted or rejected the BP, the auction automatically enters the bidding phase. Alternatively, as in eBay's Buy-it-Now auctions, we could fix the overall length of the auction at T (unless the BP is accepted). We could also consider alternative forms for the bidding phase, for example by explicitly modeling eBay's proxy bidding system. We begin with our stylized setup because it simplifies the equilibrium analysis, in part by eliminating certain equilibria, and leads more directly to identification results. In Section 5, we consider identification under some of these alternative BP auction designs and are able to extend our results under additional assumptions.

Consider a bidder who arrives at time t with a valuation v for the object. We assume that if this bidder wins the object at time t^* and pays price p^*, she obtains payoff

$$\delta(t^* - t)U(v - p^*),$$

where $U(\cdot)$ is a utility function and $\delta(\cdot)$ is a function capturing impatience, i.e. the idea that a bidder would prefer to win the object earlier. If the bidder does not obtain the object, she obtains utility 0. $\lambda(\cdot)$, $F_V(\cdot)$, $U(\cdot)$ and $\delta(\cdot)$ are primitives of our model; the variables (p, r, τ) characterize the auction setup. We make the following assumptions on these primitives.

Assumption 1 *The model primitives* $\{\lambda(\cdot), F_V(\cdot), U(\cdot), \delta(\cdot)\}$ *satisfy:*

1. $\{\lambda(\cdot), F_V(\cdot), U(\cdot), \delta(\cdot)\}$ *are common knowledge to all potential bidders;*

[4] In contrast, under Gallien and Gupta's "permanent buyout option" scheme, the option to purchase the good at price p remains for the entire duration of the auction. This was used by Yahoo! on their now defunct auction site.

2. $\lambda(\cdot)$ *is twice continuously differentiable and satisfies* $0 < \lambda(t) < \infty$ *for all* $t \geq 0$*;*

3. $F_V(\cdot)$ *is twice continuously differentiable on* $[0, \infty)$. $F_V(0) = 0$, $F_V(v) > 0$ *for all* $v > 0$, $\int_0^\infty F_V(v)dv = 1$*;*

4. $U(\cdot)$ *is twice continuously differentiable;*

5. $U'(\cdot) > \epsilon$ *for some* $\epsilon > 0$*;*

6. $-C < U''(\cdot) \leq 0$ *for some* $0 < C < \infty$ *(weak risk-aversion);*

7. $\delta'(\cdot) < 0$ *(strict impatience),* $\delta(\cdot) > 0$*;*

8. $U(0) = 0$, $U'(0) = 1$, $\delta(0) = 1$ *(normalizations).*[5]

We additionally make the following assumption about bidder behavior.

Assumption 2 *Bidders do not play weakly dominated strategies.*

The bidding phase is essentially a second-price sealed bid auction, which generally have unusual equilibria in weakly dominated strategies (Milgrom (1981), Plum (1992), Blume and Heidhues (2004)). Assumption 2 is a simple way to rule these unusual equilibria out, and it ensures that we get unique equlibrium play in the bidding phase where bidders follow the weakly dominant strategy of submitting bids equal to their valuations (or not submitting a bid if $v < r$).

With these assumptions, we can state the following

Proposition 1 *Under Assumptions 1 and 2, any symmetric, pure strategy, perfect Bayesian-Nash equilibrium (BNE) of this auction game has the following properties:*

1. *Potential bidders with* $v < r$ *never take any action;*

2. *Potential bidders with* $v > r$ *who arrive during the buy price phase* ***immediately*** *either a) "Accept the BP," i.e. purchase the good at p; or b) "Reject the BP" by placing a sealed bid equal to v;*

3. *Potential bidders with* $v > r$ *who arrive during the bidding phase place a sealed bid equal to v at some point before the end of the auction.*

Proof: *Appendix A.* □

[5] Our conditions on U are location-scale normalizations that maintain the uniqueness of the utility function up to affine transformations.

Part 2 of the Proposition states that in equilibrium, potential bidders with $v > r$ arriving during the buy price phase *do not wait*—they either accept or reject the BP immediately. The incentives to act immediately in equilibrium arise from two sources. First, waiting delays the time at which the bidder may potentially win the item, generating lower utility due to impatience. Second, waiting engenders more competition from other potential bidders for the object. For example, delaying accepting the BP incurs the risk that another potential bidder will enter and accept the BP first. Delaying rejecting the BP lengthens the time until the end of the auction (since the bidding phase has fixed length τ), increasing the expected number of competitors that the bidder will face in the sealed-bid auction.[6]

Part 3 (combined with Part 2) of the Proposition implies that we get the well known 2nd-price sealed bid auction outcome for any auction that enters the bidding phase. Specifically, the bidder with the highest valuation wins the object at the valuation of the second highest bidder, or r when there are no other bids placed.[7]

2.1 Optimal BP Decision and Conditions for a Cutoff Equilibrium

Proposition 1 does not fully characterize the BNE, as it does not specify whether a potential bidder arriving during the buy price phase (with $v > r$) will accept or reject the BP. We now characterize this decision. Consider such a bidder who arrives at t. If the bidder accepts the BP option immediately, she will obtain payoff

$$U^A(v,p) := U(v-p).$$

If the bidder rejects the BP and places a sealed bid at time t, then she will win the object if she has placed the highest sealed bid by time $t+\tau$, and pay a price equal to the valuation of the next highest bidder (if there is another bid), or equal to r if there are no other bids placed. Let

$$\gamma = \int_t^{t+\tau} \lambda(s)ds,$$

so the number of other bidders who arrive after t is Poisson(γ). (Note that γ is a function of t and τ, but we suppress this in the notation.) Then the bidder's expected utility from rejecting the BP

[6] Most parts of the proof of Proposition 1 are fairly straightforward based on this intuition. However, for Part 2.a) of the Proposition, there may be realizations of opponent arrival times and valuations such that, *ex post*, it would have been better to wait to accept the BP (e.g. while waiting, an opposing bidder with a low v bids, no other bidders enter, and the first entrant wins the item at a price that is lower than the BP). Proving that waiting to accept the BP cannot be an equilibrium strategy requires computing an expectation over the distribution of opponents' arrival times and valuations. In order to establish these results in our continuous-time setting with no upper bound on the number of opponents, the proof establishes a point process notation for the game.

[7] Our results would hold under other models of the bidding phase (e.g. if in the bidding phase, bidders could see other bidders' bids) as long as the bidding stage has the feature that the highest valuation bidder wins the item at the valuation of the second highest valuation bidder.

is

$$U^R(v,r,\tau,t) := \delta(\tau)\cdot\left\{e^{-\gamma}U(v-r)+\sum_{n=1}^{\infty}\frac{\gamma^n e^{-\gamma}}{n!}F_V^n(v)E_n\left[U(v-\max\{r,Y\}|Y\le v\right]\right\},$$

where $F_V^n(v) = [F_V(v)]^n$ and E_n is the expectation when Y has CDF F^n. In this formulation, n represents the number of other bidders that arrive after the BP is rejected, and Y represents the maximum of these other bidders' valuations. $U^R(v,r,\tau,t)$ does not depend on p, because Proposition 1 implies that any bidder arriving prior to t with $v > r$ would have already either accepted or rejected the BP. Hence, a bidder who arrives while the BP is still available knows, in equilibrium, that no prior arriving bidder has $v > r$.

The following proposition provides a simpler expression for $U^R(v,r,\tau,t)$ which we will use extensively in the sequel.[8]

Proposition 2

$$U^R(v,r,\tau,t) = \delta(\tau)\left(\alpha(r,\tau,t)U(v-r)+\int_r^v U(v-y)h(y,\tau,t)dy\right),$$

where

$$\alpha(r,\tau,t) = \exp(\gamma F_V(r)-\gamma),$$
$$h(y,\tau,t) = \exp(\gamma F_V(y)-\gamma)\gamma f_V(y),$$

and $\alpha(r,\tau,t)$, $h(y,\tau,t)$ *satisfy:*

$$\alpha(r,\tau,t)+\int_r^\infty h(y,\tau,t)dy = 1, \quad \text{and} \quad \frac{\partial\alpha(r,\tau,t)}{\partial r} = h(r,\tau,t).$$

Proof: *Online Appendix A.* $\square$

From the perspective of a bidder rejecting the BP at t, $\alpha(r,\tau,t)$ is the probability that no other bidder will arrive during the bidding phase with a valuation greater than r, and $h(y,\tau,t)$ is the density of the maximum of the valuations of bidders who arrive during the bidding phase.

With expressions for $U^A(v,p)$ and $U^R(v,r,\tau,t)$ in hand, we can now characterize the choice of whether to accept or reject the BP. In particular, we examine conditions under which this decision

[8] The simple characterizations of $\alpha(r,\tau,t)$ and $h(y,\tau,t)$ arise from the properties of the Poisson distribution. We presume these or similar results are known to researchers in point process theory, but we were unable to find the result in the literature, so we provide a proof in the Appendix.

depends on a bidder's valuation v in a particularly simple way: the bidder accepts the BP option if her valuation is above a cutoff value and rejects the BP (i.e. initiates bidding) otherwise. We call this a "cutoff strategy."

Given Proposition 1, bidders with $v > r$ who arrive during the buy price phase immediately either accept or reject the BP. Clearly, in any BNE, the bidder must accept the BP if and only if

$$U^A(v,p) \geq U^R(v,r,\tau,t)$$

or

$$U(v-p) \geq \delta(\tau)\left(\alpha(r,\tau,t)U(v-r) + \int_r^v U(v-y)h(y,\tau,t)dy\right).$$

Define

$$M(v,r,\tau,t) = U^{-1}\left(\delta(\tau)\left(\alpha(r,\tau,t)U(v-r) + \int_r^v U(v-y)h(y,\tau,t)dy\right)\right), \quad \forall r, v \geq r, t, \tau.$$

$M(v,r,\tau,t)$ is the certainty equivalent of the random outcome obtained by rejecting the BP option. Whether or not the BP decision follows a cutoff strategy depends on how the certainty equivalent varies with v. Consider the following assumption:

Assumption 3 *Let $M_v(v,r,\tau,t)$ denote the partial derivative of $M(v,r,\tau,t)$ with respect to v. For some $\epsilon > 0$, $M_v(v,r,\tau,t) < 1 - \epsilon, \quad \forall r, v \geq r, t, \tau$.*

This is a sufficient condition for equilibrium BP decisions to follow cutoff strategies.[9]

Proposition 3 *Under Assumptions 1, 2, and 3, in any symmetric, pure strategy, perfect BNE, there exists a finite valued cutoff function $c(p,r,\tau,t)$, implicitly defined by the equation*

$$U(c(p,r,\tau,t)-p) = \delta(\tau)\left(\alpha(r,\tau,t)U(c(p,r,\tau,t)-r) + \int_r^{c(p,r,\tau,t)} U(c(p,r,\tau,t)-y)h(y,\tau,t)dy\right), \tag{1}$$

such that a potential bidder who arrives at t during the buy price phase with $v > r$ immediately accepts the BP if $v > c(p,r,\tau,t)$, immediately rejects the BP if $v < c(p,r,\tau,t)$, and is indifferent between immediately accepting and immediately rejecting the BP if $v = c(p,r,\tau,t)$. The cutoff function $c(p,r,\tau,t)$ satisfies:

1. *$c_p(p,r,\tau,t) > 0$, $c_r(p,r,\tau,t) < 0$, $c_\tau(p,r,\tau,t) < 0$;*

[9] A necessary condition for the accept/reject decision to follow a cutoff strategy for any p, r, t, and τ is that $M_v(v,r,\tau,t) \leq 1$.

2. $c(r,r,\tau,t) = r$*,* $c(p,r,\tau,t) > p$ *when* $p > r$*.*

Proof: *Appendix A.* $\square$

Assumption 3 is a high level assumption. In general, whether or not it holds will depend on the forms of $U(\cdot)$, $\alpha(r,\tau,t)$, $h(y,\tau,t)$, and $\delta(\tau)$. Some intuition can be obtained in the the case where the bidder is risk neutral, i.e. $U(x) = x$. Then we have

$$M(v,r,\tau,t) = \delta(\tau)\left(\alpha(r,\tau,t)(v-r) + \int_r^v (v-y)h(y,\tau,t)dy\right),$$

$$M_v(v,r,\tau,t) = \delta(\tau)\left(\alpha(r,\tau,t) + \int_r^v h(y,\tau,t)dy\right) < \delta(\tau) < 1, \quad \forall t,\tau,v \geq r,$$

since $\alpha(r,\tau,t) + \int_r^v h(y,\tau,t)dy < 1$, $\delta(0) = 1$, and $\delta'(\tau) < 0$. Hence under risk neutrality, equilibria always involve cutoff strategies, regardless of the forms of $\alpha(r,\tau,t)$, $h(y,\tau,t)$, and $\delta(\tau)$. The intuition is fairly clear in the risk neutral case. When a bidder's valuation v increases from v^* to v^*+1, $U^A(v,p)$ increases by 1. On the other hand, $U^R(v,r,\tau,t)$ increases by less than 1 because of discounting, and because some of the utility gains from the valuation increase are lost to competing bidders with valuations between v^* and v^*+1. Since the utility from accepting the BP option increases in v faster than the utility from rejecting the BP option, optimization implies a cutoff rule where bidders with high valuations accept the BP, and bidders with low valuations reject the BP.

It is possible to obtain more primitive conditions ensuring an equilibrium in cutoff strategies. For example, in Online Appendix B we show that $U'''(x) \leq 0$ is a sufficient condition for Assumption 3 to hold for any primitives $\{\lambda(\cdot), F_V(\cdot), \delta(\cdot)\}$ satisfying our conditions. However, given a particular $\{\lambda(\cdot), F_V(\cdot), \delta(\cdot)\}$, there will generally be utility functions that do not satisfy $U'''(x) \leq 0$ but do satisfy Assumption 3.

Regarding the properties of $c(p,r,\tau,t)$, it is intuitive that when the BP p increases, the cutoff increases, making a bidder less likely to accept the BP. When the reserve price r increases, the cutoff decreases because the expected utility from rejecting the BP decreases. When τ increases, the expected utility from rejecting the BP decreases, and the cutoff decreases. There are two reasons for this. First, increasing τ increases the expected number of competitors entering in the bidding phase, lowering the expected utility from rejecting the buy price. Second, the expected utility from rejecting the BP decreases as τ increases due to impatience. Property 2 simply states that when the BP exactly equals the reserve price, all entering bidders with $v > r$ will accept the BP. As p increases above r, $c(p,r,\tau,t)$ also increases, and is strictly above p. Lastly, note that $c_t(p,r,\tau,t)$ may be positive or negative (or 0), depending on how the Poisson rate $\lambda(t)$ varies across t.

2.2 Inverse cutoff function $p(c, r, \tau, t)$

Given that the cutoff function $c(p, r, \tau, t)$ is strictly increasing in p, we can invert it to obtain an inverse cutoff function $p(c, r, \tau, t)$. The inverse cutoff function tells us, for a given r, τ, and t, what the BP would have to be for a bidder with valuation c to be indifferent between accepting and rejecting the BP. Following equation (1), the inverse cutoff function solves

$$U(c - p(c, r, \tau, t)) = \delta(\tau) \left(\alpha(r, \tau, t) U(c - r) + \int_r^c U(c - y) h(y, \tau, t) dy \right). \tag{2}$$

Unlike the cutoff function, we can explicitly solve out for the inverse cutoff function as a function of model primitives, i.e.

$$p(c, r, \tau, t) = c - U^{-1} \left(\delta(\tau) \left(\alpha(r, \tau, t) U(c - r) + \int_r^c U(c - y) h(y, \tau, t) dy \right) \right). \tag{3}$$

The inverse cutoff function and this alternative representation of the indifference condition will be useful in the identification arguments below. The following properties will also be helpful:

Proposition 4 *Under Assumptions 1, 2, and 3, the inverse cutoff function $p(c, r, \tau, t)$ satisfies the following properties:*

1. $0 < p_c(c, r, \tau, t) < 1$, $p_r(c, r, \tau, t) > 0$, $p_\tau(c, r, \tau, t) \geq 0$;
2. $r \leq p(c, r, \tau, t) \leq c$, $p(c, r, \tau, t) = c$ *iff* $c = r$;
3. $p_{cc}(c, r, \tau, t)$, $p_{rr}(c, r, \tau, t)$, *and* $p_{cr}(c, r, \tau, t)$ *exist and are bounded away from* ∞ *and* $-\infty$;
4. *Let z denote a dummy argument. Then*
   $p_c(z, z, \tau, t) = 1 - \delta(\tau)\alpha(z, \tau, t)$,
   $p_r(z, z, \tau, t) = \delta(\tau)\alpha(z, \tau, t)$,
   $p_{cc}(z, z, \tau, t) = -U''(0)\, \delta(\tau)\alpha(z, \tau, t)\, (1 - \delta(\tau)\alpha(z, \tau, t)) - \delta(\tau) h(z, \tau, t)$,
   $p_{rr}(z, z, \tau, t) = -U''(0)\, \delta(\tau)\alpha(z, \tau, t)\, (1 - \delta(\tau)\alpha(z, \tau, t)) + \delta(\tau)\alpha'(z, \tau, t)$,
   $p_{cr}(z, z, \tau, t) = U''(0)\delta(\tau)\alpha(z, \tau, t)\, (1 - \delta(\tau)\alpha(z, \tau, t))$.

Proof: *Online Appendix A.* $\square$

Property 4 in this proposition concerns the behavior of the inverse cutoff function when the buy price (and thus the cutoff) equals the reserve price (in which case all arriving bidders with $v > r$ accept the BP). Since we assume that $p \geq r$ (and thus $c \geq r$), the derivatives when $c = r$ should be interpreted as one-sided derivatives.

3 Identification

We now consider identification of the structural demand parameters $\{F_V(\cdot), \lambda(\cdot), U(\cdot), \delta(\cdot)\}$ of this model. Heuristically, we suppose we have many independent observations of auctions with the same $\{F_V(\cdot), \lambda(\cdot), U(\cdot), \delta(\cdot)\}$, and exogenous variation in the reserve price (r), buy price (p), and bidding phase length (τ). Formally, we define random variables R, P, Υ, whose realizations are r, p, τ. Let $F_{r,p,\tau}$ denote their joint distribution. Conditional on $R = r, P = p, \Upsilon = \tau$, we have a distribution for the auction outcomes determined by the (fixed) structural demand parameters, and the auction mechanism and equilibrium solution described in Section 2. Given knowledge of the joint distribution of R, P, Υ and the auction outcomes, we want to recover the structural demand parameters.

The assumption that variation in the reserve price, buy price, and length is exogenous may be strong in some situations. Even if we view the identification analysis as conditional on auction-level covariates, the variation in r, p, and τ could arise from unobserved (to the econometrician) differences across auctions that sellers take into account when choosing the auction features (Krasnokutskaya (2004), Asker (2008), Roberts (2013), Descarolis (2009)). It may be possible to relax this exogeneity assumption using instrumental variables techniques, but we do not pursue this in the current paper. However, the assumption may be credible in some markets where one believes the majority of variation in auction setup is due to seller characteristics rather than unobserved demand factors, or in lab or field experiments with randomized variation in the auction features (and as mentioned in Section 1, some recent studies show evidence of sellers experimenting on eBay).

Our single-unit theoretical model in Section 2 also embodies an assumption that there are no other auctions, either simultaneously, or in the future. This is often assumed in the auction literature (with some notable exceptions, e.g. Jofre-Bonet and Pesendorfer (2003), Zeithammer (2006, 2007, 2009), Nekipelov (2008), Zeithammer and Adams (2010), and Backus and Lewis (2009)), but it may be questionable for some product categories on eBay and similar markets.

3.1 Observational Model

Now we specify the auction outcomes that are observed. Let T_1 be the time of the first action (either accepting or rejecting the BP) taken by any bidder in the auction. Let $B = 0$ indicate that the first acting bidder rejected the BP, and let $B = 1$ indicate that the first acting bidder accepted the BP. The parameters of the model $\{F_V(\cdot), \lambda(\cdot), U(\cdot), \delta(\cdot)\}$ determine a joint distribution for (T_1, B) given $(P = p, R = r, \Upsilon = \tau)$. Let $F_1(\cdot|p, r, \tau)$ denote the conditional distribution of T_1 given $(P = p, R = r, \Upsilon = \tau)$, and let $\Pr(B = 1|p, r, \tau, t_1)$ denote the conditional probability of the BP option being accepted given $(P = p, R = r, \Upsilon = \tau, T_1 = t_1,)$.

Our basic identification results will only require that the outcomes T_1 and B are observed (along with the "exogenous" variables ($P = p, R = r, \Upsilon = \tau$)). This will be enough to identify the structural parameters $\{F_V(\cdot), \lambda(\cdot), U(\cdot), \delta(\cdot)\}$, using the implications of Propositions 1 and 3. In principle, we might observe other outcome variables, for example the final price in the auction, or the sealed bids placed by participants, or the proxy bids in eBay auctions.[10] In Sections 4 and 5 we examine the identifying power of some of these other outcome variables.

To derive the simplest version of our identification results, we make the following assumption on the support of (R, P, Υ):

Assumption 4 *The marginal distribution of R has support $[0, \infty)$ and the conditional distribution of P given $R = r$ has support $[r, \infty)$. The conditional distribution of Υ given $(R = r, P = p)$ has support $[0, \infty)$.*

This large support condition is relaxed in Section 4.

3.2 Identification of $\lambda(\cdot)$ and $F_V(\cdot)$

We begin by examining identification of the arrival rate and valuation distribution. Our arguments for identification of these two objects are similar to Canals-Cerda and Pearcy (2013), who consider identification in eBay auctions without buy prices (and without impatience or risk aversion). Recall that potential bidders arrive according to a Poisson process with arrival rate $\lambda(t)$, and by Proposition 1, if no other action has yet been taken, the arriving bidder takes an action if her valuation $V \geq r$. Hence the time of the first observed action T_1 in an auction, given $(P = p, R = r, \Upsilon = \tau)$, has conditional hazard rate

$$\theta(t_1|p, r, \tau) = \lambda(t_1)(1 - F_V(r)). \tag{4}$$

To separately identify $\lambda(\cdot)$ and $F_V(\cdot)$, note that when $r = 0$, $F_V(r) = 0$, so

$$\lambda(t_1) = \theta(t_1|p, 0, \tau).$$

Since the Poisson intensity is bounded, $T_1|P, R = 0$ has full support $[0, \infty)$, so conditioning on $R = 0$ identifies $\lambda(\cdot)$ on $[0, \infty)$. Put simply, when the reserve price is zero, every potential bidder who arrives takes an action, so we can recover the arrival rate of bidders.

Given that $\lambda(\cdot)$ has been identified, we can identify $F_V(\cdot)$ by using equation (4) for values of r other than 0. This identifies $F_V(\cdot)$ on $[0, \infty)$.

Since $\lambda(\cdot)$ and $F_V(\cdot)$ are identified, it follows that $\alpha(r, \tau, t_1)$ and $h(y, \tau, t_1)$ are identified over their

[10] In some cases, these other variables may be hard to interpret, e.g. eBay's proxy bids.

full supports, since we can form $\gamma = \int_{t_1}^{t_1+\tau} \lambda(s)ds$ for any t_1 and then directly construct

$$\alpha(r,\tau,t_1) = \exp(\gamma F_V(r) - \gamma),$$
$$h(y,\tau,t_1) = \exp(\gamma F_V(y) - \gamma)\gamma f_V(y).$$

3.3 Identification of $c(p,r,\tau,t_1)$ and $p(c,r,\tau,t_1)$

To identify the cutoff function and its inverse, we use the observed distribution of B given $(P = p, R = r, \Upsilon = \tau, T_1 = t_1)$. For a bidder who takes an action at time t_1, the distribution of her valuation V is F_V truncated below at r. By Proposition 3, this bidder will accept the buy price if $V \geq c(p,r,\tau,t_1)$. Therefore,

$$\Pr(B = 1|p,r,\tau,t_1) = \frac{1 - F_V(c(p,r,\tau,t_1))}{1 - F_V(r)}.$$

Given knowledge of F_V and the conditional probability of $B = 1$, we can invert to obtain

$$c(p,r,\tau,t_1) = F_V^{-1}\left(1 - (1 - F_V(r))\Pr(B = 1|p,r,\tau,t_1)\right).$$

This identifies the cutoff function on the joint support of (P,R,Υ,T_1). Having identified $c(p,r,\tau,t_1)$, we can then invert it to obtain the inverse cutoff function $p(c,r,\tau,t_1)$ and the first and second derivatives of $p(c,r,\tau,t_1)$ with respect to c and r.

3.4 Identification of $U(\cdot)$ and $\delta(\cdot)$

We now consider identification of the utility and impatience functions. Our approach is related to the general identification strategy of Chiappori et al (2009), although they only consider identification of a utility function without a time-preference component. As in their paper, we have a one-dimensional form of unobserved heterogeneity (the bidder valuations v), and the implication of our model that bidders use cutoff strategies amounts to a single-crossing property. Chiappori et al (2009) propose to use indifference conditions to recover $U(\cdot)$. In our situation with $U(\cdot)$ and $\delta(\cdot)$, this indifference condition is

$$U(c(p,r,\tau,t_1)-p) = \delta(\tau)\left(\alpha(r,\tau,t_1)U(c(p,r,\tau,t_1) - r) + \int_r^{c(p,r,\tau,t_1)} U(c(p,r,\tau,t_1) - y)h(y,\tau,t_1)dy\right), \tag{5}$$

where $\alpha(r,\tau,t_1), h(y,\tau,t_1)$ and $c(p,r,\tau,t_1)$ have already been identified from the arguments above. This integral equation will hold for all $t_1 \in [0,\infty)$, $r \in [0,\infty)$, $p \in [r,\infty)$, and $\tau \in (0,\infty)$. We need to show that there is a unique utility function $U(\cdot)$ and impatience function $\delta(\cdot)$ that satisfy this equation. To simplify the discussion, fix $t_1 = t_1^0$ and $\tau = \tau^0$, resulting in

$$U(c(p,r)-p) = \delta\left(\alpha(r)U(c(p,r)-r) + \int_r^{c(p,r)} U(c(p,r)-y)h(y)dy\right) \quad \forall r \in [0,\infty), p \in [r,\infty). \tag{6}$$

Now the question is whether there is a unique function $U(\cdot)$ and scalar $\delta = \delta\left(\tau^0\right)$ satisfying this indifference condition. (Throughout this section, uniqueness of U will mean uniqueness up to affine transformations.) Note that if this can be shown to be the case, it can be also be done for other values of τ to identify the function $\delta(\cdot)$ over the support of τ.

3.4.1 General Formulation of Problem

We start by relating our identification question to existing literature on certainty equivalents. This provides a general overview of the identification arguments necessary to separately identify the utility and impatience functions. We then show how the specific variation in p and r in our BP situation leads to an identification result. For illustration purposes, first suppose there is no impatience, i.e. $\delta = 1$. Then Equation (6) matches certain outcomes (of the form $c(p,r) - p$) to lotteries with distributions determined by the functions α and h and the auction setup p and r. A general form of the problem is the following: given a collection of probability distributions $\mathcal{F}$ and a certainty equivalence functional $M : \mathcal{F} \to \mathbb{R}$, is there a unique (up to affine transformations) utility function $U(\cdot)$ such that

$$M(F) = U^{-1}\left(\int U(x)dF(x)\right), \quad \forall F \in \mathcal{F}. \tag{7}$$

where in our specific case, the collection of distributions $\mathcal{F}$ is generated by the variation in auction setup p and r,[11] and the certainty equivalent function $M(F) = c(p,r) - p$.

By the Kolmogorov–Nagumo–De Finetti Theorem (De Finetti, 1931), if $\mathcal{F}$ contains all probability distributions (on a bounded interval) and M satisfies some basic conditions, there is a continuous, strictly increasing function $U(\cdot)$ satisfying (7), and $U(\cdot)$ is unique. This establishes that a wide range of certainty equivalence functionals can be rationalized (over all lotteries) by some utility function. For the purposes of identification, we know that such a utility function exists (because the certainty equivalent is generated by it) and instead want to show that a relatively small set of distributions $\mathcal{F}$ (generated by the variation in r and c) suffice to recover it uniquely from knowledge of M. In fact, inspection of a classical proof of the theorem given in Hardy, Littlewood, and Polya (1952) shows that uniqueness can be established when $\mathcal{F}$ is a certain one-dimensional class of lotteries, those generated by mixtures of the endpoints of the bounded interval supporting the lotteries. (We review this argument briefly in Online Appendix C.) This suggests that, at least when there is no impatience, a simple class of lotteries may be enough to trace out the utility

[11] Note that F will have a point mass at the point r and its support will be bounded at $c(p,r)$.

function from its certainty equivalents, if the variation across lotteries is of a suitable form.

Identification become more complicated when we incorporate impatience. We now have

$$M(F) = U^{-1}\left(\delta \int U(x)dF(x)\right), \quad \forall F \in \mathcal{F}. \tag{8}$$

where δ also needs to be identified. Fortunately, in our setting the BP lotteries are indexed by *two* parameters, p and r. Suppose that we have two sets of lotteries $\mathcal{F}_1 = \{F_1(x;\theta_1)\}$ and $\mathcal{F}_2 = \{F_2(x,\theta_2)\}$ indexed by scalar parameters θ_1 and θ_2. Suppose that the parametrizations are such that θ_1 and θ_2 are equal to the certainty equivalents of the corresponding lottery. Then we have

$$\begin{aligned} U(\theta_1) &= \delta \int U(x)dF_1(x;\theta_1) \\ U(\theta_2) &= \delta \int U(x)dF_2(x;\theta_2) \end{aligned}$$

Each of these equations is a homogeneous Fredholm equation. For a given value of δ, one of these two equations would typically suffice to recover U up to normalization, and in some cases U could be recovered by standard methods for solving Fredholm equations. However, when δ is unknown, a single equation would not generally suffice, but the richer class of lotteries $\mathcal{F}_1 \cup \mathcal{F}_2$ may allow one to recover both U and δ. Verifying this is likely to be possible only on a case-by-case basis. Next, we show that the variation in lotteries generated by θ_1 and θ_2 (i.e p and r) in our BP setting is sufficient to identify both $U(\cdot)$ and δ. Moreover we show that the particular manner in which p and r affect the kernels of the two Fredholm equations allows us to recover U by solving a simple first order differential equation, and we can also obtain a closed form expression for the Arrow-Pratt measure of risk aversion as a function of the observable data distribution.

3.4.2 BP Identification

To show identification of $U(\cdot)$ and $\delta(\cdot)$ in our BP model, it is helpful to express the integral equation indifference condition in the space of the inverse cutoff function rather than the cutoff function (recall, that the inverse cutoff function $p(c,r,\tau,t)$ gives the BP p that would make a bidder with valuation c indifferent between accepting or rejecting the BP). As noted in Section 2, this indifference condition can be written as

$$U(c - p(c,r,\tau,t_1)) = \delta(\tau)\left(\alpha(r,\tau,t_1)U(c-r) + \int_r^c U(c-y)h(y,\tau,t_1)dy\right). \tag{9}$$

This integral equation will hold for all $t_1 \in [0,\infty)$, $r \in [0,\infty)$, $c \in [r,\infty)$, and $\tau \in (0,\infty)$. To identify the utility function $U(\cdot)$, we need to show that given knowledge of $\alpha(r,\tau,t_1), h(y,\tau,t_1)$ and $p(c,r,\tau,t_1)$, there is a unique utility function $U(\cdot)$ that satisfies this integral equation.

We can differentiate the integral equation with respect to c:

$$U'\left(c - p(c,r,\tau,t_1)\right)\left(1 - p_c(c,r,\tau,t_1)\right) = \delta(\tau)\left(\alpha(r,\tau,t_1)U'(c-r) + \int_r^c U'(c-y)h(y,\tau,t_1)dy\right); \tag{10}$$

and with respect to r:

$$U'\left(c - p(c,r,\tau,t_1)\right)\left(-p_r(c,r,\tau,t_1)\right) = \delta(\tau)\left(-\alpha(r,\tau,t_1)U'(c-r) + U(c-r)\left(\frac{\partial\alpha(r,\tau,t_1)}{\partial r} - h(r,\tau,t_1)\right)\right); \tag{11}$$

$$U'\left(c - p(c,r,\tau,t_1)\right)\left(-p_r(c,r,\tau,t_1)\right) = -\delta(\tau)\alpha(r,\tau,t_1)U'(c-r); \tag{12}$$

where the second line follows because $\frac{\partial\alpha(r,\tau,t_1)}{\partial r} = h(r,\tau,t_1)$.

Under our assumptions, each side of (12) is bounded away from 0. Dividing (10) by (12) cancels the delay term (and eliminates the impatience term), and differentiating the resulting equation with respect to r results in an ordinary first order linear differential equation in $U'(\cdot)$, i.e.

$$U''(c-r) = \frac{\Phi_r(c,r,\tau,t_1) + h(r,\tau,t_1)}{\Phi(c,r,\tau,t_1)}U'(c-r), \tag{13}$$

where

$$\Phi(c,r,\tau,t_1) = \alpha(r,\tau,t_1)\left[\frac{(1-p_c(c,r,\tau,t_1))}{p_r(c,r,\tau,t_1)} - 1\right].$$

Intuitively, the ratio of derivatives answers the following question: for a given increase in r from r_0 to r_1 (which decreases the cutoff), what is the increase in p from p_0 to p_1 (which increases the cutoff) that would lead to the same cutoff value, i.e. the same indifferent bidder? The increase in r from r_0 to r_1 not only decreases the mean payoff from rejecting the BP but also decreases the variance of this payoff (since r truncates the distribution of outcomes from rejecting). Both of these effects are known from the arrival process and distribution of valuations which we previously identified. Thus, the difference between p_0 and p_1 (which determine the certain payoff from accepting the BP) tells us how bidders value the decreased risk moving from r_0 to r_1 (controlling for the known change in mean payoff moving from r_0 to r_1). The impatience function does not enter the above argument because it cancels from Equation (9).

Since $h(r,\tau,t_1)$ and all the components of $\Phi(c,r,\tau,t_1)$ have already been shown to be identified, we only need to consider whether there is a unique solution $U(\cdot)$ to this differential equation (with $U(0) = 0$ and $U'(0) = 1$).

Proposition 5 *Under Assumptions 1, 3, and 4, there is a unique* $U(\cdot)$ *on support* $[0,\infty)$ *satisfying (13). Hence,* $U(\cdot)$ *is identified on support* $[0,\infty)$.

Proof: *Appendix A.* □

First order linear differential equations like (13) typically have a unique solution given an initial condition (which in our case is $U'(0) = 1$). Note that (13) holds for any values of $(c(p, r, \tau, t_1), r, \tau, t_1)$. So, for example, one can fix (r, τ, t_1) and use variation in p across its support (i.e. in $c(p, r, \tau, t_1)$) to trace out $U(\cdot)$. Since this can be done at any (r, τ, t_1), this will generate overidentifying restrictions, which we discuss in the next section.

Equation (13) can be rewritten as

$$\frac{U''(c-r)}{U'(c-r)} = \frac{\Phi_r(c, r, \tau, t_1) + h(r, \tau, t_1)}{\Phi(c, r, \tau, t_1)}. \tag{14}$$

As a result, to identify the Arrow-Pratt measure of risk aversion at a certain point one only needs to compute the values of $\Phi(c, r, \tau, t_1)$, $\Phi_r(c, r, \tau, t_1)$, and $h(r, \tau, t_1)$ at that point. On a related note, we investigate how sensitive our identification results are to the support condition (Assumption 4) in Section 4.1.

Lastly, consider identification of the impatience function. The impatience function does not enter the above argument because it cancels from Equation (9). However, given identification of U, δ can be identified from the level of the cutoff, because a higher cutoff implies more patience.

Manipulating the indifference condition (9) gives us

$$\delta(\tau) = \frac{U(c - p(c, r, \tau, t_1))}{\alpha(r, \tau, t_1)U(c-r) + \int_r^c U(c-y)h(y, \tau, t_1)dy}. \tag{15}$$

Since all of the terms on the right-hand-side have already been shown to be identified, it is clear that $\delta(\cdot)$ is identified. Since τ varies over support $(0, \infty)$ and we have normalized $\delta(0) = 1$, $\delta(\cdot)$ is identified on $[0, \infty)$. As noted above, the precise way that p and r vary the auction lottery are key to this fairly simple proof of identification (and the closed form representation of the Arrow-Pratt measure of risk aversion). For example, note that in (9), r only enters through the point mass term and the limit of integration. This means that the integral disappears when differentiating w.r.t. r, and allows the simple cancellation of the impatience term when taking the ratio of the derivatives.

4 Extensions

4.1 Relaxing Support Conditions

Assumption 4 requires that the exogenous random variables (R, P, Υ) have large supports, which may be unrealistic in many applications. We now examine how restrictions on the supports affect our identification results. First consider a situation where all the auctions in the data set have a fixed bidding phase length τ_0.

Assumption 5 *The marginal distribution of R has support $[0, \infty)$ and the conditional distribution of P given $R = r$ has support $[r, \infty)$. The conditional distribution of Υ given (R, P) has $Pr(\Upsilon = \tau_0 | R, P) = 1$ almost surely.*

Proposition 6 *Under Assumptions 1, 3, and 5, $\{F_V(\cdot), \lambda(\cdot), U(\cdot)\}$ are identified over their full supports, and $\delta(\tau)$ is identified at τ_0.*

Intuitively, if the bidding phase has fixed length τ_0, we can only identify $\delta(\cdot)$ at that point. However, the other structural functions are identified over their entire supports by our previous arguments.

In practice, it may also be difficult to estimate $\lambda(t)$ for large t, because this would require observing many auctions that last until time t without an action being taken. To capture this situation, we suppose that observations on auctions are truncated at some time $\overline{T}$:

Assumption 6 *There is some $\overline{T} > \tau_0$ such that we only observe (T_1, B) when $T_1 < \overline{T}$.*

In this case, even though we can only identify the Poisson process prior to time $\overline{T}$, we can still identify $F_V(\cdot)$ and $U(\cdot)$. Specifically, the following result follows from the arguments in Section 3.

Proposition 7 *Under Assumptions 1, 3, 5, and 6, $\{F_V(\cdot), U(\cdot)\}$ are identified over their full supports, $\lambda(\cdot)$ is identified on support $[0, \overline{T})$, and $\delta(\tau)$ is identified at τ_0.*

Next, we consider restricting the support of R to the bounded set $[\underline{r}, \overline{r}]$.

Assumption 7 *The marginal distribution of R has support $[\underline{r}, \overline{r}]$ and the conditional distribution of P given $R = r$ has support $[r, \infty)$. The conditional distribution of Υ given (R, P) has $Pr(\Upsilon = \tau_0 | R, P) = 1$ almost surely. There is some $\overline{T} > \tau_0$ such that we only observe (T_1, B) when $T_1 < \overline{T}$.*

Proposition 8 *Under Assumptions 1, 3, and 7, $\lambda(t)(1 - F_V(r))$ is identified on $(r, t) \in [\underline{r}, \overline{r}] \times [0, \overline{T})$, $\delta(\tau)$ is identified at τ_0, and $U(\cdot)$is identified on $[0, \overline{r} - \underline{r}]$.*

Proof: *Online Appendix A.* $\square$

In this case, since the reserve price never goes below $\underline{r}$, we cannot distinguish between a non-arrival and an arrival of a bidder with valuation below $\underline{r}$. As a consequence, $\lambda(t)$ and $F_V(r)$ are not separately identified. However, over limited supports, we can identify the composite function $\lambda(t)(1 - F_V(r))$, and through this $\alpha(r, \tau, t_1), h(y, \tau, t_1)$ and $c(p, r, \tau, t_1)$.[12] A slight modification

[12]What prevents separate identification of $\lambda(t)$ and $F_V(r)$ is the lower bound $\underline{r}$. This is only restrictive if one wants to consider alternative auction setups with reserve prices below $\underline{r}$. If the reserve price is above $\underline{r}$, it is not important to distinguish nonarrival of bidders from arrival of from bidders with valuations below $\underline{r}$.

of our original identification argument leads to the final result. The support on which $U(\cdot)$ is identified depends on the range of the reserve price in the support of the data.

Lastly, we further restrict the support of the buy price P.

Assumption 8 *The marginal distribution of R has support $[\underline{r}, \overline{r}]$ and the conditional distribution of P given $R = r$ has support containing $[p_0 - \epsilon, p_0 + \epsilon]$ for some $\epsilon > 0$. The conditional distribution of Υ given (R, P) has $Pr(\Upsilon = \tau_0 | R, P) = 1$ almost surely. There is some $\overline{T} > \tau_0$ such that we only observe (T_1, B) when $T_1 < \overline{T}$. There exists $r^* \in (\underline{r}, \overline{r})$ and $t^* < \overline{T} - \tau_0$ such that $c(p_0, r^*, \tau_0, t^*) \in (\underline{r}, \overline{r})$.*

Proposition 9 *Under Assumptions 1, 3, and 8, $\lambda(t)(1 - F_V(r))$ is identified on $(r, t) \in [\underline{r}, \overline{r}] \times [0, \overline{T})$, $\delta(\tau)$ is identified at τ_0, and $\frac{U''(\cdot)}{U'(\cdot)}$ is identified at the point $c(p_0, r^*, \tau_0, t^*) - r^*$.*

Proof: *Online Appendix A.* $\square$

The last condition of Assumption 8 requires the observed buy price p_0 to be low enough so that the cutoff at this buy price is within the range $[\underline{r}, \overline{r}]$ (for some r^* and t^*). This is needed to identify the cutoff function. However, provided this holds, we only need a small amount of variation in the buy price to identify the Arrow-Pratt measure of risk aversion at a particular $c(p_0, r^*, \tau_0, t^*) - r^*$.[13] If there is a set of points (r^*, t^*) such that $r^* \in (\underline{r}, \overline{r})$, $t^* < \overline{T} - \tau_0$, and $c(p_0, r^*, \tau_0, t^*) \in (\underline{r}, \overline{r})$, then the Arrow-Pratt measure of risk aversion will be identified at all the corresponding values of $c(p_0, r^*, \tau_0, t^*) - r^*$.

In summary, we can relax our original support conditions in various ways and still obtain "local" identification of the structural objects. However, even Assumption 8 makes a significant joint support condition on r and p. Intuitively, to identify $U(\cdot)$ and $\delta(\cdot)$ locally, we need variation in the reserve price and we need that there are buy prices low enough that the equilibrium cutoff is sometimes in this range.

4.2 Optimal Auction Design with a Limited Support and Additional Parametric Assumptions

While the structural parameters of the model may be of independent interest, one reason for estimating them is to learn about the implications of the model for optimal auction design. In the current context, this design would include the reserve price (r), buy price (p), and bidding phase length (τ), and a natural question is what values of (r, p, τ) maximize expected revenue to the seller.

[13]This implies, for example, that if bidders are assumed to have a CARA utility function, we could identify the CARA coefficient with only local variation in the buy price.

Clearly, if the structural parameters $\{F_V(\cdot), \lambda(\cdot), U(\cdot), \delta(\cdot)\}$ are fully identified, it is conceptually simple to find the (r, p, τ) that maximize expected revenue (though in practice this may require extensive computations). However, this is not possible in general when the structural parameters are only partially identified due to limited supports as in Section 4.1. Intuitively, one needs to know the utility and impatience functions at *all* points to calculate expected revenue for all (r, p, τ).

Nevertheless, the results in Section 4.1 can still be helpful to the auction designer. Specifically, if one is willing to impose additional parametric assumptions, it may be possible to determine the optimal auction design even with a limited support.[14] For example, Proposition 9 implies that if one further assumes that the impatience function is geometric (i.e. $\delta(\tau) = e^{-\delta\tau}$) and the utility function is CARA (i.e. $U(x) = 1 - e^{-\alpha x}$), then the local variation in Assumption 8 fully identifies both $U(\cdot)$ and $\delta(\cdot)$ (and therefore α and δ). The same would hold if $U(\cdot)$ and $\delta(\cdot)$ were assumed to belong to other families of single parameter functions (e.g. CRRA). If there is more variation in Υ and P than the very local form in Assumptions 5 and 8, one could also use more flexible (i.e. more than a single parameter) specifications for $U(\cdot)$ and $\delta(\cdot)$. The term $\lambda(t)(1 - F_V(r))$ could also be parametrically specified as a function of t and r, and this would be identified given that Proposition 9 implies that $\lambda(t)(1-F_V(r))$ is identified on $(r, t) \in [\underline{r}, \overline{r}] \times [0, \overline{T})$. Thus, under additional parametric assumptions, the entire model is identified even with limited support conditions, and one can assess various aspects of optimal auction design. Lastly, even if one does not attempt to fully identify $U(\cdot)$ and $\delta(\cdot)$ by making parametric assumptions, the above results imply that one could test for either the existence of risk aversion or impatience (using only the limited variation in the above assumptions). Finding evidence of either one, even without fully identifying the model, could provide evidence to a seller or auction designer that buy price auctions might increase revenue over standard auctions.

4.3 Testable Restrictions

Our model has a number of testable restrictions on the observed data (or functions of the observed data). These follow from the discussion in Sections 2 and 3 and include:

1. The conditional hazard $\theta(t_1|p, r, \tau)$ does not depend on p or τ;
2. The ratio $\frac{\theta(t_1|p,r,\tau)}{\theta(t_1|p,0,\tau)}$ does not depend on p,τ, or t_1;
3. $r \le p(c, r, \tau, t_1) \le c$, $0 < p_c(c, r, \tau, t_1) < 1$, $p_r(c, r, \tau, t_1) > 0$, $p_\tau(c, r, \tau, t_1) > 0$, $p_c(r, r, \tau, t_1) = 1 - \alpha(r, \tau, t_1)$, $p_r(r, r, \tau, t_1) = \alpha(r, \tau, t_1)$;

[14] Aradillas-Lopez, Gandhi, and Quint (2013) have an elegant result in an ascending auction framework with correlated private values where even though the underlying structural parameters (the distribution of values) is not identified, one can place bounds on seller profit (and potentially the optimal reserve price), *without* any additional parametric assumptions. It does not appear that such an approach would be fruitful in our context, perhaps because of the additional two structural parameters $U(\cdot)$ and $\delta(\cdot)$ and the additional two policy parameters p and τ.

4. The coefficient in the differential equation (13), i.e.

$$\frac{\Phi_r(c,r,\tau,t_1)+h(r,\tau,t_1)}{\Phi(c,r,\tau,t_1)},$$

does not depend on τ and t_1. The coefficient only depends on c and r through the difference $c-r$.[15]

These restrictions could be tested, and also imply that more flexible versions of the model can be identified. For example, restrictions 1 and 2 imply that one can identify a model where the distribution of a bidder's valuation depends on their time of arrival, i.e. $F_V(v,t)$. Restriction 4 implies we could identify an extended model where a bidder's utility function depends on their time of arrival, i.e. $\delta(\tau)U\left(v-p,t\right)$.[16] Ideally, we would like to find necessary and sufficient conditions on $\theta(t_1|p,r,\tau)$ and $\Pr(B=1|p,r,\tau,t_1)$ (which can be estimated directly from data) for these to be rationalized by our model $\{F_V(\cdot),\lambda(\cdot),U(\cdot),\delta(\cdot)\}$ (see, e.g. Aryal, Perrigne, and Vuong (2016)). However, this appears to be quite complicated in our context.For example, the inverse cutoff function (3) implies that the relationship between $p(c,r,\tau,t_1)$ (and thus $\Pr(B=1|p,r,\tau,t_1)$) at two values of t_1 depends in a complicated way on the arrival process between those two points in time (as an integrand through $h(r,\tau,t_1)$ and then through the inverse utility function).

4.4 Additional Data on Final Prices

Our basic identification argument in Section 3 only uses data on T_1, the time of the first observed arrival, and B, the indicator for whether the BP option was accepted or rejected. This data identifies the arrival rate $\lambda(t)$, the valuation distribution $F_V(v)$, and the inverse cutoff equation $p(c,r,\tau,t_1)$. Using $\lambda(t)$ and $F_V(v)$, we can identify the functions $\alpha(r,\tau,t_1)$ and $h(y,\tau,t_1)$ in our integral equation (5). Given knowledge of $\alpha(r,\tau,t_1)$, $h(y,\tau,t_1)$, and $p(c,r,\tau,t_1)$, we then showed identification of the utility components $U\left(\cdot\right)$ and $\delta\left(\cdot\right)$.

Perhaps the most surprising aspect of this is that we use *no* data on either bids or final transaction prices. In this section, we investigate conditions under which final transaction prices can strengthen our identification results. We illustrate conditions under which data on final prices allow one to further weaken the support conditions described in Section 4.1. More specifically, we weaken Assumption 8 to the following:

[15] The fact that (13) implies

$$\frac{U''(c-r)}{U'(c-r)}=\frac{\Phi_r(c,r,\tau,t_1)+h(r,\tau,t_1)}{\Phi(c,r,\tau,t_1)}$$

generates these restrictions.

[16] One could potentially investigate whether even more general models are identified or partially identified, e.g. the utility function $U(\tau,v-p,t)$ or $U\left(\tau,v,p,t\right)$, or models where bidders are heterogeneous in their risk attitudes or their impatience rather than in their valuations. Even more challenging would be models where bidder heterogeneity cannot be summarized by a scalar quantity.

Assumption 9 *The marginal distribution of* R *has support* $[\underline{r}, \overline{r}]$ *and the conditional distribution of* P *given* $R = r$ *has support containing* $[p_0 - \epsilon, p_0 + \epsilon]$ *for some* $\epsilon > 0$. *The conditional distribution of* Υ *given* (R, P) *has* $Pr(\Upsilon = \tau_0 | R, P) = 1$ *almost surely. There is some* $\overline{T} > 0$ *such that we only observe* (T_1, B) *when* $T_1 < \overline{T}$. *There exists* $r^* \in (\underline{r}, \overline{r})$ *and* $t^* < \overline{T} - \tau_0$ *such that* $c(p_0, r^*, \tau_0, t^*) \in (\underline{r}, \overline{r})$.

The difference between Assumption 8 and Assumption 9 is the support of T_1. Assumption 8 requires that we observe arrivals (and BP) decisions up to time $T_1 > \tau$. This ensures that we are able to identify enough of $\lambda(t)(1 - F_V(r))$ (which incorporates the arrival rate and distribution of values) such that we can identify $\alpha(r, \tau, t_1)$ and $h(y, \tau, t_1)$ for at least some t_1 in the data. Intuitively, to identify the level of competition a bidder will expect to face if they reject the BP at time t_1, we need to know $\lambda(t)(1 - F_V(r))$ over the entire bidding phase (and hence we need $T_1 > \tau$).

Assumption 9 is weaker because it does not require $T_1 > \tau$. Recall that $\alpha(r, \tau, t_1)$ and $h(y, \tau, t_1)$ measure the level of competition a bidder will expect to face if they reject the BP at time t_1. In the previous sections, $\alpha(r, \tau, t_1)$ and $h(y, \tau, t_1)$ were constructed using $\lambda(t)(1 - F_V(r))$, which was identified from arrival patterns later in the auction. Instead, as we show below, we can use realized final prices from those auctions to identify $\alpha(r, \tau, t_1)$ and $h(y, \tau, t_1)$. This may be particularly helpful if the data do not include many auctions that have no activity until later in the auction.

This alternative approach requires two additional assumptions.

Assumption 10 *We observe final transaction prices* Z *and the identity of the winning bidder (specifically, we can identify if the bidder who rejected the BP is the bidder who wins the auction)*

Assumption 11 *If an auction enters the bidding phase, the final price in the auction is the 2nd highest valuation of all bidders who arrived at the auction.*

The first assumption requires that we observe final transaction prices, and that we know whether the winner of the auction was the bidder who rejected the BP. The second assumption, which follows Haile and Tamer (2003) and Canals-Cerda and Pearcy (2013), holds in equilibrium if the bidding phase consists of either a 2nd price-sealed bid auction, a button auction, or an eBay style proxy-bidding auction (up to bid increments).

Now, consider an auction with setup (p, r, τ). Suppose that at $T_1 = t_1 < \overline{T}$, an arriving bidder rejects the BP option ($B = 0$). This bidder has value V distributed according to F_V truncated between r and $c(p, r, \tau, t_1)$. Let $\tilde{Y}$ equal the highest valuation among bidders arriving *after* time t_1, or equal R if no further bidders arrive to the auction with valuations greater than r. Under our assumptions, $\tilde{Y}$ is conditionally independent of V, i.e.

$$\tilde{Y} \perp V | P = p, R = r, \Upsilon = \tau, T_1 = t_1, B = 0.$$

Suppose we observe the random variable W, an indicator that the bidder who rejected the BP option ended up winning the auction. Note that $W = 1(\tilde{Y} < V)$, since the bidder who rejected the BP only wins if her valuation is higher than the bidders entering during the bidding phase.

Assumption 10 implies that we observe the final price in the auction, Z, conditional on $W = 1$. Under Assumption 11, $Z = \tilde{Y}$ if $W = 1$. In other words, if the bidder who rejected the BP option wins the auction, then the final price Z is equal to the highest valuation of bidders arriving after time t_1 (or r if there are no such bidders with valuations greater than r). The proposition below states that, given the above data set up, we can identify aspects of the utility function and the time-discounting function.

Proposition 10 *Under Assumptions 1, 3, 9, 10 and 11,* $\lambda(t)(1 - F_V(r))$ *is identified on* $(r,t) \in [\underline{r}, \overline{r}] \times [0, \overline{T})$, $\delta(\tau)$ *is identified at* τ_0, *and* $\frac{U''(\cdot)}{U'(\cdot)}$ *is identified at the point* $c(p_0, r^*, \tau_0, t^*) - r^*$.

Proof: *Appendix A.* □

This identification argument first uses the data on W and Z to recover the conditional probability of $W = 1$ and the distribution of Z given $W = 1$. Then we can identify $\alpha(r, \tau, t_1)$ which is the probability that, given rejection of the BP at t_1, no other bidder with valuation greater than r arrives during the bidding phase, and $h(y, \tau, t_1)$, the density of the maximum valuation of all future bidders arriving during the bidding phase (or the reserve price).[17] Once we have $\alpha(r, \tau, t_1)$ and $h(y, \tau, t_1)$, we can use equation (5) to identify aspects of $U(\cdot)$ and $\delta(\cdot)$[18]

This approach allows us to identify aspects of $U(\cdot)$ and $\delta(\cdot)$ without having to identify $\lambda(t)(1 - F_V(r))$ at later points in the auction. This approach is therefore less demanding in that we do not need to observe auctions that last a long time before any observed arrivals, and could be particularly useful for our extension to eBay "Buy-It-Now" auctions considered below. Note that this approach also provides additional testable restrictions of the model. For example, $p_y(y|r, p, \tau, t_1, B = 0)$ should not depend on p.

The identification proposed in this section only uses the final price conditional on the BP rejector winning the auction. In principle, there is more information that could be used for identifying structural objects and generating testable restrictions of our model. For example, we could use the final price regardless of who wins the auction. If we also observe all the bids in the bidding phase, and these bids represent bidders' valuations, then we could use this information as well.[19] If we also observe clickstream data measuring when a user first visited a particular auction, this could provide an alternative source of identification for arrival rates.

[17]In equilibrium, no bidder who arrived prior to t_1 has valuation $> r$.

[18]The approach detailed in this section is related to a large literature on estimation methods for dynamic models initiated by Hotz and Miller (1993).

[19]Depending on the context, bids may only provide bounds on valuations (Haile and Tamer (2003), Zeithammer and Adams (2010)).

5 Applications

5.1 eBay's Buy-it-Now Auctions

eBay's popular Buy-it-Now auctions feature a BP option that disappears as soon as any bidder places a bid, which our original model captures. However, in eBay's BP auction format, there is a fixed length for the overall auction. Since eBay's auctions end at some fixed time T, the bidding phase has length $T - t_1$, not a fixed length τ as we assumed in our model. We call our BP auction model a "Fixed τ" BP auction, whereas eBay's Buy-it-Now auction is a "Fixed T" BP auction. In a Fixed T auction, the environment is defined by (p, r, T). The cutoff function for a bidder arriving at t_1 is then $c(p, r, T, t_1)$, depending on p, r, and the overall length of the auction T. A number of papers have empirically studied eBay (or related) Buy-it-Now auctions, including Wan, Teo, and Zhu (2003), Chan, Kadiyali, and Park (2007), and Anderson, Friedman, Milam, and Singh (2008).

Identification of parameters in a Fixed T auction is similar to that in a Fixed τ auction. Variation in reserve prices across auctions can trace out arrival rates and valuations, variation in p identifies the cutoff function $c(p, r, T, t_1)$, and an integral equation similar to (6), i.e.

$$U(c - p(c, r, T, t_1)) = \delta(T - t_1) \left(\alpha(r, T, t_1) U(c - r) + \int_r^c U(c - y) h(y, T, t_1) dy \right), \qquad (16)$$

identifies $U(\cdot)$ and $\delta(\cdot)$.

However, the Fixed T BP auction has a complication that does not arise in our original model. Consider a potential bidder ariving at some time t, with a valuation greater than r but less than $c(p, r, T, t)$. In our original model, such a bidder has an incentive to immediately reject the BP. This is because immediately rejecting ends the auction sooner, and minimizes the expected amount of competition that bidder faces in the bidding phase. In contrast, in a Fixed T model, this bidder *does not* have a strict incentive to immediately reject the BP, since immediately rejecting does not end the auction sooner or limit competition.[20] This can lead to multiple equilibria, because bidders are indifferent between rejecting the BP immediately, or waiting.[21] Then we may not be able to identify $F_V(r)$, since we are not certain that observed actions are being taken by bidders who have just arrived.[22]

We can resolve this problem by either modifying the model, or using a solution concept that ensures

[20] Moreover, there is no reason to act immediately to prevent another bidder from entering and accepting the BP. This is because in a model where bidders are only heterogeneous in their valuations, this other bidder would always win the auction phase.

[21] In the Fixed T environment, bidders with valuations *above* the cutoff $c(p, r, T, t_1)$ do have an incentive to accept the BP immediately, as they do not want to lose the item to another arriving bidder.

[22] This point illustrates an interesting theoretical advantage of Fixed τ BP auctions vs Fixed T BP auctions. Specifically, identification is easier in Fixed τ auctions because of the stronger incentives for bidders to act immediately. It would be interesting to compare expected revenue across the two types of BP auctions, though that seems beyond the scope of the current paper.

that bidders who reject the BP act immediately. Gallien and Gupta (2007) discuss restricting attention to trembling hand perfect BNE, where the trembles involve a bidder accidentally accepting the BP. This results in an equilibrium where bidders who want to reject the BP do so immediately.[23] The same is true if one adds a small cost to the model that is incurred when one waits (or needs to return to the auction at a later point) to reject the BP.[24] In both cases, our identification arguments in Sections 2 and 3 can be applied to Fixed T auctions.

Because we need to ensure that certain subsets of the bidders act immediately upon entering the auction,[25] we are also relying heavily on the assumption that auctions are isolated. If the same (or a similar) good were potentially available in other eBay auctions, then bidders might have an incentive to wait before taking an action. This suggests that it might be valuable to investigate how our identification arguments extend to dynamic multiple-auction settings (e.g. Jofre-Bonet and Pesendorfer (2003), Zeithammer (2006), Backus and Lewis (2009)).[26]

While our extension to eBay's Buy-it-Now auctions require some further strong assumptions about behavior of bidders, our approach to identification has a number of attractive features. First, recall that in Fixed τ auctions, if τ is fixed in the data at τ_0, one can identify $\delta(\cdot)$ only at τ_0. In a Fixed T auction, even if T is fixed in the data at T_0, one can identify the impatience function $\delta(\cdot)$ at more than one point, due to variation in t_1. Second, our identification arguments do not require data on eBay proxy bids. As noted by Zeithammer and Adams (2010) among others, these bids may be hard to interpret on eBay. We investigate many of these and other issues in our ongoing empirical work on eBay's Buy-it-Now auctions (Ackerberg, Hirano, and Shahriar, 2006).

5.2 GMAC Buy Price Auctions

GMAC uses a type of BP auction to sell fleet cars (cars coming off lease) to auto dealers around the US. In these auctions, the seller (GMAC) sets a BP p and a reserve price r. There are three distinct phases of the auction. In the first phase, only the option to buy the car at p is available to bidders—they cannot place bids. After a fixed length of time $\widetilde{T}$, the auction enters the second phase, in which bidders can either accept the BP or reject the BP by placing a regular bid. If any bidder rejects the BP, the BP disappears for all bidders, and the auction enters the third phase,

[23] Consider a bidder who arrives with a valuation greater than r, but less than $c(p, r, T, t)$ (i.e. he prefers to reject the BP). If the bidder waits, there is some probability that another bidder with a lower valuation (also greater than r) will tremble and accept the BP (this other bidder should optimally reject the BP). This generates a strict incentive for the original bidder to reject the BP immediately.

[24] In another model in Gallien and Gupta (2007), there is assumed to be a point mass of "desperate" bidders who are very impatient and accept the BP immediately if they arrive. This also creates incentives for normal bidder to reject the BP immediately. However, with this model, one would want to explicitly consider identification of the point mass of desperate bidders, and check whether this affects identification of the other model components. The same issue arises in a perturbation where one adds a small utility benefit of participating in the bidding phase.

[25] We could extend the argument to allow for an exogenous random delay before acting.

[26] In eBay auctions it may also be hard to identify the arrival process late in the auction, because this requires data on auctions in which no action is taken until close to the end of the auction. Hence the alternative approach described in Section 4.3 to identify $\alpha(r, T, t_1)$ and $h(y, T, t_1)$ using final prices may be especially useful here.

where bidders can only place bids (using a proxy bidding system similar to eBay's). The auction ends at a fixed point in time T.

Thus, GMAC fleet auctions are similar to eBay auctions, except they have an introductory phase of fixed length, in which bidders cannot place regular bids, and in which the BP continues to be available unless it is accepted. The setup of a GMAC auction is described by $(p, r, T, \widetilde{T})$.

As with eBay auctions, bidders who plan to reject the BP do not have strict incentives to do so immediately upon arrival. (Bidders who arrive at $t_1 < \widetilde{T}$ cannot immediately reject the BP even if they want to.) We could use similar arguments to the eBay case to ensure that bidders act immediately (when they can do so). However, we can avoid these arguments by making use of the initial phase of the GMAC auction.

Consider a GMAC auction where $p = r$. For such auctions, *all* potential bidders have strict incentives to act immediately. In any pure strategy, symmetric BNE, all arriving bidders with valuations greater than $p = r$ immediately accept the BP, because waiting risks the possibility that another bidder will accept the BP instead.[27]

Thus, if the distribution of $P|R = r$ has positive probability of $P = r$ for $r \in [\underline{r}, \overline{r}]$, we can identify $\lambda(t)(1 - F_V(r))$ on $[\underline{r}, \overline{r}]$ for all t. This is because the hazard rate of the first observed action, $\theta(t_1|p, r, T, \widetilde{T})$, satisfies

$$\theta(t_1|p, r, T, \widetilde{T}) = \lambda(t_1)(1 - F_V(r)) \quad \text{when } p = r. \tag{17}$$

To identify the cutoff function $c(p, r, T, \widetilde{T}, t_1)$, we need to also consider auctions where $p > r$. When $p \geq r$, the hazard rate of the first observed action satisfies

$$\theta(t_1|p, r, T, \widetilde{T}) = \lambda(t_1)(1 - F_V(c(p, r, T, \widetilde{T}, t_1))) \quad \text{for } t_1 < \widetilde{T}. \tag{18}$$

Note that (18) *only* holds when $t_1 < \widetilde{T}$. At $\widetilde{T}$ and after, first actions can also be taken by bidders who arrived earlier and have been "waiting." This complicates matters for $t_1 \geq \widetilde{T}$, because the hazard rate of first observed action then depends on equilibrium selection issues (i.e. how long people wait). Fortunately, we will be able to identify parts of the functions $U(\cdot)$ and $\delta(\cdot)$ using only data when $t_1 < \widetilde{T}$.

We now show that $c(p, r, T, \widetilde{T}, t_1)$ is identified over limited domains. When $t_1 < \widetilde{T}$, the cutoff function $c(p, r, T, \widetilde{T}, t_1)$ can be shown to satisfy the implicit equation

$$\theta(t_1|c(p, r, T, \widetilde{T}, t_1), c(p, r, T, \widetilde{T}, t_1), T, \widetilde{T}) = \theta(t_1|p, r, T, \widetilde{T}).$$

Intuitively, this says that the hazard rate in an auction at $(t_1, p, r, T, \widetilde{T})$ is equivalent to the hazard

[27] Note that an auction with $p = r$ is analagous to a posted price sales mechanism. On eBay there are a non-trivial number of auctions with $p = r$, but we do not know whether this also occurs in GMAC auctions.

rate in a hypothetical auction where the reserve price and buy price are both set at $c(p, r, T, \widetilde{T}, t_1)$. The hazard on the left hand side of the equation, i.e. $\theta(t_1|k, k, T, \widetilde{T})$ is strictly decreasing in k, and identified in the data between $\theta(t_1|\overline{r}, \overline{r}, T, \widetilde{T})$ and $\theta(t_1|\underline{r}, \underline{r}, T, \widetilde{T})$. The hazard on the right-hand-side is identified over its entire support in the data. Hence, $c(p, r, T, \widetilde{T}, t_1)$ is identified as long as $\theta(t_1|\overline{r}, \overline{r}, T, \widetilde{T}) < \theta(t_1|p, r, T, \widetilde{T}) < \theta(t_1|\underline{r}, \underline{r}, T, \widetilde{T})$. Intuitively, we can identify the cutoff function at all points where the cutoff is between $\underline{r}$ and $\overline{r}$. Note that this implies that the inverse cutoff function $p(c, r, T, \widetilde{T}, t_1)$ is identified on $c \in [\underline{r}, \overline{r}]$. Recall that these results only hold when $t_1 < \widetilde{T}$.

Next, we need to identify $\alpha(r, T, \widetilde{T}, t_1)$ and $h(y, T, \widetilde{T}, t_1)$ in the integral equation

$$U(c - p(c, r, T, \widetilde{T}, t_1)) = \delta(T - t_1) \left(\alpha(r, T, \widetilde{T}, t_1) U(c - r) + \int_r^c U(c - y) h(y, T, \widetilde{T}, t_1) dy \right). \quad (19)$$

The right-hand side measures the expected utility if the bidder does not accept the BP, but the functions $\alpha(r, T, \widetilde{T}, t_1)$ and $h(y, T, \widetilde{T}, t_1)$ are now more complicated. Bidders arriving at $t_1 < \widetilde{T}$ know that there may be a set of bidders who arrived beforehand that may participate in the bidding phase. Moreover, this set of prior arriving bidders have valuation distributions truncated between r and the cutoff function (which depends on the time[28]). However, given that $\lambda(t)(1 - F_V(r))$ is already identified over all t and $r \in [\underline{r}, \overline{r}]$, and given that $c(p, r, T, \widetilde{T}, t_1)$ is already identified for all $t < \widetilde{T}$ and on domain $[\underline{r}, \overline{r}]$, one can show that $\alpha(r, T, \widetilde{T}, t_1)$ is identified on $r \in [\underline{r}, \overline{r}]$ and that $h(y, T, \widetilde{T}, t_1)$ is identified on $y \in [\underline{r}, \overline{r}]$. This allows one to identify $\delta(\cdot)$ on $[T, T - \widetilde{T})$, and $U(\cdot)$ on $[0, \overline{r}]$.

To summarize, we have shown that in the GMAC fleet auction mechanism, one can identify the structural parameters, at least over certain domains. In contrast to the eBay example, we did not have to perturb the model to incentivize BP rejectors to act immediately. The introductory period in GMAC fleet auctions is useful because in this period, we know that all BP rejectors must wait to place a bid. The problem is more severe on eBay, because we cannot observe how many BP rejectors wait, and how many act immediately.

6 Conclusion

A BP auction allows a bidder to avoid the risk of losing the auction and to obtain the item sooner. As a result, the bidder's behavior in a BP auction is affected by her risk and time preferences. The existing theoretical literature on BP auctions has shown that when bidders are known to be either impatient or risk averse, sellers can increase expected revenue by using BP auctions. Our paper takes a different perspective, focusing on the extent to which data from BP auctions allows a researcher (or seller) to identify bidders' risk aversion and time preferences. Using general forms

[28]Note that in this model, the equilibrium cutoff function should increase in t_1 because of 1) strict discounting, and 2) since in equilibrium, the later the BP is still available, the less competition one can anticipate in the bidding phase.

of bidders' valuation distributions, risk aversion and impatience, the paper develops an IPV model for an auction with a temporary BP. We first characterize equilibria in the model and then study identification. Given sufficient variation in auction characteristics such as the reserve price, the buy price, and the length of the bidding phase, we show that the four unknown structural objects—the arrival rate, valuation distribution, utility function, and the time-discounting function—are all nonparametrically identified.

The paper also provides extensions of the results to cases where the variation in auction characteristics is limited. We show how local identification of the structural objects can still be obtained in these cases. Under some additional assumptions, we extend our identification results to the specific setup of eBay's Buy-it-Now auctions and GMAC's BP auctions for fleet cars. Our results provide some alternative options for researchers; for example we show that one can recover the structural parameters using the buy-price decision and/or final prices without requiring a model for proxy bidding in eBay style auctions. The best combination of assumptions and estimation strategies will likely be application-specific. In addition, while our identification arguments can be used "constructively" to nonparametrically estimate the auction features, other methods such as maximum likelihood or moment-matching with a flexible parametric specification may be useful when the sample size is limited. For example, Ackerberg, Hirano, and Shahriar (2006) develop a partial likelihood estimator for eBay's Buy-it-Now auctions which imposes some parametric restrictions while avoiding a detailed specification of proxy bidding.

Our paper contributes to the literature on identification in auctions, on BP auctions, and more generally to the literature on recovering risk and time preferences from observed behavior when there is unobserved heterogeneity. Possible future extensions could include endogenizing auction characteristics, allowing higher dimensional bidder heterogeneity, or considering a dynamic environment with sequential or simultaneous auctions.

A Appendix: Proofs of Propositions

A.1 Proof of Proposition 1

We first establish some notation. We can view our model as a random-player game (Milchtaich 2004), with the random set of players described by a point process. For background on point processes, see Kallenberg (1983), Fristedt and Gray (1996) Ch. 29, and Kallenberg (2010) Ch. 12.

Let $\Gamma = \mathbb{R}_+ \times \mathbb{R}_+$ be the set of possible bidder types. We interpret $\gamma = (a, v) \in \Gamma$ to mean that a player arrives at time a and has valuation v. In our model, valuations are drawn independently from a probability distribution with CDF F_v and potential bidders arrive according to arrival rate $\lambda(t)$. We also use F_v and λ to denote the corresponding measures, and define $\Pi = \lambda \times F_v$. By our assumptions, this defines a Poisson point process with intensity Π. This process selects a random set of player-types $\{(a_1, v_1), \ldots, (a_k, v_k)\}$.

By Kallenberg (1983), Theorem 11.1, a bidder who knows her own type has a posterior for her competitors that is also a Poisson point process with intensity Π. (See also Miltaich (2004).) Thus, from the point of view of any individual bidder who has arrived, her competitors are generated by the same Poisson point process.

Next, we define bidder actions and strategies. At each time t and given the history of the auction up to time t, a bidder can accept the buy price, reject the buy price and bid, or wait, and can choose the amount of the bid. Let S be the set of action functions that are feasible according to the rules of the auction. A strategy $\sigma : \Gamma \to S$ maps type into state- and time-contingent actions. We restrict attention to symmetric, pure strategies, so (in equilibrium) a single σ will represent the strategy profile of players, and will generate a collection of action profiles Σ of all the bidders. Let $P_{\Pi\sigma}$ denote the distribution of Σ induced by the distribution Π over player times and the symmetric strategy σ.

Bidder i cares about the profile Σ^{-i} which describes the state-contingent action functions of players other than i. We use the notation $\Sigma^{-i}(R)$ to denote the number of other players who play strategies belonging to the set $R \subset S$. By the conditioning properties of the Poisson process mentioned above, and the assumption that bidders' valuations are independent, the distribution of Σ^{-i} is also given by $P_{\Pi\sigma}$. Bidder i's realized utility in the auction game is given by

$$U\left(a_i, v_i, \sigma\left(a_i, v_i\right), \Sigma^{-i}\right),$$

where the specific form of the utility function could be derived from the rules of the game as described in the main text.

Consider a candidate equilibrium of this game σ. We want to show by contradiction that under our assumptions, σ must satisfy the 3 statements of the Proposition. We first show Statement 1, then Statement 3, and finally Statement 2.

Statement 1: Potential bidders with $v_i < r$ never take any action.

This is straightforward to show given that Assumption 2 rules out weakly dominated strategies.

Statement 3: Potential bidders with $v_i > r$ who arrive during the bidding phase place a sealed bid equal to v_i at some point before the end of the auction.

This is also straightforward given that Assumption 2 rules out weakly dominated strategies.

Statement 2: Potential bidders with $v_i > r$ who arrive during the buy price phase immediately either a) "Accept the BP," i.e. purchase the good at p; or b) "Reject the BP" by placing a sealed bid equal to v_i.

The proof of this statement is a bit more involved. Note that it involves decisions prior to the auction entering the bidding phase. Hence, we enforce the implications of Statements 1 and 3 on behavior in the bidding phase. In particular, if the auction ever enters the bidding phase, all bidders arriving before the end of the bidding phase with $v > r$ bid their valuations.

Consider a bidder arriving at time a_i while the BP is still available. Define

$$S_1 = \{s \in S \mid s \text{ prescribes accepting/rejecting the BP prior to } a_i\}.$$

Then, at time a_i, this bidder's expected utility is:

$$E\left[U\left(a_i, v_i, \sigma\left(a_i, v_i\right), \Sigma^{-i}\right) \mid \Sigma^{-i}\left(S_1\right) = 0, a_i, v_i\right] = \int U\left(a_i, v_i, \sigma\left(a_i, v_i\right), \Sigma^{-i}\right) dP\left(\Sigma^{-i} \mid \Sigma^{-i}\left(S_1\right) = 0\right),$$

where $P(\cdot \mid \cdot)$ is the conditional probability measure of Σ^{-i} implied by $P_{\Pi\sigma}$. This conditions on $\Sigma^{-i}\left(S_1\right) = 0$, because bidder i knows that the BP is still available, and hence the realized profile Σ^{-i} has no points in the set S_1 of state-contingent action functions under which the BP would have been accepted or rejected prior to a_i. Other than this, we do not need to fully specify anything about $P\left(\Sigma^{-i} \mid \Sigma^{-i}\left(S_1\right) = 0\right)$, except that the opponent strategies must satisfy the implications of Statements 1 and 3 with probability one.

Now suppose that σ prescribes that a bidder arriving while the BP is still available at a_i with valuation $v_i > r$ waits upon arrival at a_i (i.e. they do not immediately reject or accept the BP). We want to show that this contradicts σ being an equilibrium. We can consider three possibilities for σ, depending on what it prescribes in the *hypothetical* scenario where no other bidders take actions after time a_i:

1. if no other bidders were to take action after a_i, there is some finite $t_R^* > a_i$ such that the bidder waits until t_R^* and then rejects the BP at t_R^*;

2. if no other bidders were to take action after a_i, the bidder waits indefinitely (i.e. never accepting or rejecting the BP);

3. if no other bidders were to take action after a_i, there is some finite $t_A^* > a_i$ such that the bidder waits until t_A^* and then accepts the BP at t_A^*.

We consider these alternatives one by one, in each case contradicting σ being an optimal strategy.

(Case 1) Consider the alternative strategy $\sigma^*(a_i, v_i)$ that is identical to $\sigma(a_i, v_i)$ except that it prescribes that a bidder with (a_i, v_i) immediately rejects the BP. For any realization of Σ^{-i} (satisfying Statements 1 and 3), it must be the case that

$$U\left(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}\right) \geq U\left(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}\right).$$

This is because under $\sigma^*(a_i, v_i)$ the auction enters the bidding phase with with probability 1 at a_i, and therefore, the bidder will compete in the bidding phase with only bidders that arrive prior to $a_i + \tau$. Even if the auction ends up in the bidding phase under $\sigma(a_i, v_i)$, the bidder competes with a weakly larger set of bidders.

It is also the case that for any opponent strategies satisfying Statements 1 and 3, there are realizations of Σ^{-i} with positive probability given $\Sigma^{-i}(S_1) = 0$ such that

$$U\left(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}\right) > U\left(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}\right).$$

To find such a set, consider realizations of Σ^{-i} corresponding to cases where no other bidders arrive through time $a_i + \tau$, and one other bidder (with $v_i > v_j > r$) enters in the time interval $(a_i + \tau, t_R^* + \tau]$. Under strategy $\sigma^*(a_i, v_i)$, the bidder wins the item at r, while under strategy $\sigma(a_i, v_i)$, the bidder wins the item at v_j.

Hence, as long as opponents strategies satisfy Statements 1 and 3, it must be the case that

$$E\left[U\left(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}\right) \mid \Sigma^{-i}(S_1) = 0, a_i, v_i\right] > E\left[U\left(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}\right) \mid \Sigma^{-i}(S_1) = 0, a_i, v_i\right],$$

which contradicts any strategy involving waiting to reject the BP being an equilibrium.

(Case 2) Consider the alternative strategy $\sigma^*(a_i, v_i)$ that is identical to $\sigma(a_i, v_i)$ except that it prescribes that a bidder with (a_i, v_i) immediately rejects the BP (rather than waiting indefinitely, as prescribed by $\sigma(a_i, v_i)$). For any opponent strategies (satisfying Statements 1 and 3) and any realization of Σ^{-i}, it must be the case that

$$U\left(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}\right) \geq U\left(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}\right),$$

by arguments identical to Case 1.

It is also the case that for any opponent strategies satisfying Statements 1 and 3, there are realizations of Σ^{-i} with positive probability given $\Sigma^{-i}(S_1) = 0$ such that

$$U\left(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}\right) > U\left(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}\right).$$

To find such a set, consider realizations of Σ^{-i} corresponding to cases where no other bidders arrive through time $a_i + \tau$, and one other bidder (with $v_j > v_i > r$) enters (and takes an action) after time $a_i + \tau$. Under strategy $\sigma^*(a_i, v_i)$, the bidder wins the item at r, while under strategy $\sigma(a_i, v_i)$, the bidder never wins the item.

Hence, as long as opponents strategies satisfy Statements 1 and 3, it must be the case that

$$E\left[U\left(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}\right) \mid \Sigma^{-i}(S_1) = 0, a_i, v_i\right] > E\left[U\left(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}\right) \mid \Sigma^{-i}(S_1) = 0, a_i, v_i\right],$$

which contradicts any strategy involving waiting indefinitely being an equilibrium.

(Case 3) Now consider two alternative strategies: $\sigma^*(a_i, v_i)$, where a realized bidder with (a_i, v_i) immediately rejects the BP, and $\sigma^{**}(a_i, v_i)$, where a realized bidder with (a_i, v_i) immediately accepts the BP. We will show that either

$$E\left[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0\right] > E\left[U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i} \mid \Sigma^{-i}(S_1) = 0\right],$$

or

$$E\left[U(a_i, v_i, \sigma^{**}(a_i, v_i), \Sigma^{-i} \mid \Sigma^{-i}(S_1) = 0\right] > E\left[U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i} \mid \Sigma^{-i}(S_1) = 0\right],$$

which contradicts σ being an equilibrium.

Define

$$S_2 = \{s \in S \mid s \text{ prescribes accepting/rejecting the BP at or prior to } t_A^*\}.$$

We can then write

$$\begin{aligned}
&E\left[U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i} \mid \Sigma^{-i}(S_1) = 0\right] \\
&= \Pr\left(\Sigma^{-i}(S_2) = 0 \mid \Sigma^{-i}(S_1) = 0\right) E\left[U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_2) = 0, \Sigma^{-i}(S_1) = 0\right] \\
&+ \Pr\left(\Sigma^{-i}(S_2) > 0 \mid \Sigma^{-i}(S_1) = 0\right) E\left[U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_2) > 0, \Sigma^{-i}(S_1) = 0\right].
\end{aligned} \quad (20)$$

First, note that

$$\begin{aligned}
&E\left[U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_2) = 0, \Sigma^{-i}(S_1) = 0\right] \\
&= \delta(t_A^* - a_i)U(v_i - p) \\
&< U(v_i - p) \\
&= E\left[U(a_i, v_i, \sigma^{**}(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0\right]. \qquad (21)
\end{aligned}$$

The strict inequality follows from strict discounting. The last equality follows because according to $\sigma^{**}(a_i, v_i)$, the bidder immediately accepts the BP and obtains $U(v_i - p)$ (this assumes either that 1) there is a tiebreaking rule that if more than one bidder accepts/rejects at exactly t, the action of the most recent arrival takes precedence, or 2) competitor strategies are such that another bidder accepting or rejecting the BP at exactly a_i is a zero probability event).

Next, note that

$$\begin{aligned}
&E\left[U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_2) > 0, \Sigma^{-i}(S_1) = 0\right] \\
&\leq E\left[U(a_i, v_i, \sigma^{*}(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_2) > 0, \Sigma^{-i}(S_1) = 0\right] \\
&\leq E\left[U(a_i, v_i, \sigma^{*}(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0\right]. \qquad (22)
\end{aligned}$$

The first inequality follows because for any opponent strategies satisfying Statements 1 and 3, for any realization of Σ^{-i} such that $\Sigma^{-i}(S_2) > 0$ and $\Sigma^{-i}(S_1) = 0$, it must be the case that

$$U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}) \leq U(a_i, v_i, \sigma^{*}(a_i, v_i), \Sigma^{-i}).$$

This is because for any realization of Σ^{-i} under which bidder i wins the auction using strategy $\sigma(a_i, v_i)$, the bidder would also win the auction (at a weakly lower price) under $\sigma^{*}(a_i, v_i)$ (since under $\sigma^{*}(a_i, v_i)$ the auction enters the bidding phase immediately). The second inequality of (22) follows from Lemma 10 below.

Combining (20), (21), and (22), we have

$$\begin{aligned}
&E\left[U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0\right] \\
&< \Pr\left(\Sigma^{-i}(S_2) = 0 \mid \Sigma^{-i}(S_1) = 0\right) E\left[U(a_i, v_i, \sigma^{**}(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0\right] \\
&+ \Pr\left(\Sigma^{-i}(S_2) > 0 \mid \Sigma^{-i}(S_1) = 0\right) E\left[U(a_i, v_i, \sigma^{*}(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0\right].
\end{aligned}$$

Since $\Pr\left(\Sigma^{-i}(S_2) = 0 \mid \Sigma^{-i}(S_1) = 0\right) + \Pr\left(\Sigma^{-i}(S_2) > 0 \mid \Sigma^{-i}(S_1) = 0\right) = 1$, it must be the case that either

$$E\left[U(a_i, v_i, \sigma^{**}(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0\right] > E\left[U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0\right],$$

or

$$E\left[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0\right] > E\left[U(a_i, v_i, \sigma(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0\right].$$

Hence, σ cannot be an equilibrium. □

Lemma 11 *Under the conditions of Proposition 1,*

$$E\left[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_2) > 0, \Sigma^{-i}(S_1) = 0\right] \leq E\left[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0\right].$$

Proof: It suffices to show that

$$E\left[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_2) = 0, \Sigma^{-i}(S_1) = 0\right] \geq E\left[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0\right]$$

Intuitively, the expected utility from immediately rejecting the BP is higher when we condition on the realized Σ^{-i} not having opponents who would have accepted or rejected the BP prior to t_A^*. Divide the random point measure Σ^{-i} into two components, $\Sigma_{S_2}^{-i}$ and $\Sigma_{\sim S_2}^{-i}$. $\Sigma_{S_2}^{-i}$ is the restriction of Σ^{-i} to S_2, and $\Sigma_{\sim S_2}^{-i}$ is the restriction of Σ^{-i} to the complement of S_2. By the Poisson property, realizations of these point measures are independent of each other, so

$$\begin{aligned}
&E\left[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_2) = 0, \Sigma^{-i}(S_1) = 0\right] \qquad (23)\\
&= E\left[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_2) = 0\right]\\
&= \int U\left(a_i, v_i, \sigma^*(a_i, v_i), \Sigma_{S_2}^{-i}, \Sigma_{\sim S_2}^{-i}\right) dP\left(\Sigma_{\sim S_2}^{-i}\right) dP\left(\Sigma_{S_2}^{-i} \mid \Sigma_{S_2}^{-i}(S_2) = 0\right)\\
&= \int U\left(a_i, v_i, \sigma^*(a_i, v_i), \{0\}, \Sigma_{\sim S_2}^{-i}\right) dP\left(\Sigma_{\sim S_2}^{-i}\right).
\end{aligned}$$

The first equality follows because $S_1 \subseteq S_2$. The second equality follows from the independence of the two random point measures. The third equality follows because $\Sigma_{S_2}^{-i}(S_2) = 0$ implies that $\Sigma_{S_2}^{-i}(S) = 0$ for any set $S \subseteq S_2$. Therefore, conditioning on $\Sigma_{S_2}^{-i}(S_2) = 0$ makes $\Sigma_{S_2}^{-i}$ non-random. (Our notation denotes this realization of $\Sigma_{S_2}^{-i}$ as $\{0\}$.) Similarly,

$$\begin{aligned}
&E\left[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0\right] \qquad (24)\\
&= \int U\left(a_i, v_i, \sigma^*(a_i, v_i), \Sigma_{S_2}^{-i}, \Sigma_{\sim S_2}^{-i}\right) dP\left(\Sigma_{\sim S_2}^{-i}\right) dP\left(\Sigma_{S_2}^{-i} \mid \Sigma_{S_1}^{-i}(S_1) = 0\right)\\
&\leq \int U\left(a_i, v_i, \sigma^*(a_i, v_i), \{0\}, \Sigma_{\sim S_2}^{-i}\right) dP\left(\Sigma_{\sim S_2}^{-i}\right).
\end{aligned}$$

The last inequality follows because as long as opponents strategies satisfy Statements 1 and 3,

$$U\left(a_i, v_i, \sigma^*(a_i, v_i), \{0\}, \Sigma_{\sim S_2}^{-i}\right) \geq U\left(a_i, v_i, \sigma^*(a_i, v_i), \Sigma_{S_2}^{-i}, \Sigma_{\sim S_2}^{-i}\right),$$

for any realization of $\Sigma_{S_2}^{-i}$ and $\Sigma_{\sim S_2}^{-i}$ (i.e. the bidder is weakly better off with weakly fewer competitors in the auction). Combining (23) and (24), we get our desired result:

$$E\left[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_2) = 0, \Sigma^{-i}(S_1) = 0\right] \geq E\left[U(a_i, v_i, \sigma^*(a_i, v_i), \Sigma^{-i}) \mid \Sigma^{-i}(S_1) = 0\right].$$

□

A.2 Proof of Proposition 2

See Online Appendix A.

A.3 Proof of Proposition 3

The certainty equivalent function $M(v, r, \tau, t)$ is given by

$$M(v, r, \tau, t) = U^{-1}\left(\delta(\tau)\left(\alpha(r, \tau, t)U(v - r) + \int_r^v U(v - y)h(y, \tau, t)dy\right)\right).$$

We want to show that given Assumptions 1, 2, and 3, there exists a cutoff function $c(p, r, \tau, t)$ such that when $v > r$:

1. For $v > c(p, r, \tau, t)$, $U^A(v, p) > U^R(v, r, \tau, t)$;
2. For $v < c(p, r, \tau, t)$, $U^A(v, p) < U^R(v, r, \tau, t)$;
3. For $v = c(p, r, \tau, t)$, $U^A(v, p) = U^R(v, r, \tau, t)$.

First, suppose that $p = r$. In this case,

$$\begin{aligned}
U^R(v, r, \tau, t) &= \delta(\tau)\left(\alpha(p, \tau, t)U(v - p) + \int_p^v U(v - y)h(y, \tau, t)dy\right) \\
&< \delta(\tau)U(v - p) \\
&< U(v - p) \\
&= U^A(v, p),
\end{aligned}$$

for all $v > r$. The strict inequalities follow from the conditions on F_V, λ, U, and δ in Assumption 1. Hence, in this case we can define the cutoff function $c(p, r, \tau, t) = r$, since all bidders with $v > r$ strictly prefer accepting the BP to rejecting it (and those with $v = r$ are indifferent).

Now, consider a buy price $p > r$ and define $c(p, r, \tau, t)$ to be the c that solves

$$G(c, p, r, \tau, t) := c - p - M(c, r, \tau, t) = 0. \tag{25}$$

We show that such a c exists and is unique and finite. First note that if $c = r$, $G(c,p,r,\tau,t) = r - p < 0$ (since $M(r,r,\tau,t) = 0$ and $p > r$).

Next, Assumption 3 implies that

$$\frac{\partial G(c,p,r,\tau,t)}{\partial c} = 1 - M_v(c,r,\tau,t) > \epsilon > 0.$$

Hence, at $c = r + \frac{p-r}{\epsilon}$, $c - p - M(c,r,\tau,t) > 0$. Since $M(c,r,\tau,t)$ can be shown to be continuous and strictly increasing in its first argument, there is a unique $c \in (r, r + \frac{p-r}{\epsilon})$ s.t. $c - p - M(c,r,\tau,t) = 0$. By the global implicit function theorem (e.g. Ge and Wang (2002), Lemma 1), the function $c(p,r,\tau,t)$ exists.

Since $G(c,p,r,\tau,t)$ is strictly increasing in c, it must be the case that for $v > c(p,r,\tau,t)$,

$$\begin{aligned} v - p - M(v,r,\tau,t) > 0; \\ v - p > M(v,r,\tau,t); \\ U(v-p) > \delta(\tau)\left(\alpha(r,\tau,t)U(v-r) + \int_r^v U(v-y)h(y,\tau,t)dy\right); \end{aligned}$$

so potential bidders with $v > c(p,r,\tau,t)$ will optimally accept the BP. Similarly, when $v < c(p,r,\tau,t)$,

$$\begin{aligned} v - p - M(v,r,\tau,t) < 0; \\ v - p < M(v,r,\tau,t); \\ U(v-p) < \delta(\tau)\left(\alpha(r,\tau,t)U(v-r) + \int_r^v U(v-y)h(y,\tau,t)dy\right); \end{aligned}$$

so the potential bidder will optimally reject the BP. Potential bidders with $v = c(p,r,\tau,t)$ are indifferent between accepting and rejecting the BP. Hence, under our assumptions, the BP decision follows a cutoff strategy.

Since $c(p,r,\tau,t) - M(c(p,r,\tau,t),r,\tau,t) = p$ for $p > r$, we also have

$$\begin{aligned} c(p,r,\tau,t) - p &= M(c(p,r,\tau,t),r,\tau,t); \\ U\left(c(p,r,\tau,t) - p\right) &= U\left(M(c(p,r,\tau,t),r,\tau,t)\right); \\ U\left(c(p,r,\tau,t) - p\right) &= \delta(\tau)\left(\alpha(r,\tau,t)U(c(p,r,\tau,t) - r) + \int_r^{c(p,r,\tau,t)} U(c(p,r,\tau,t) - y)h(y,\tau,t)dy\right). \end{aligned}$$

Note that this equation is also satisfied when $p = r$ (and thus $c(p,r,\tau,t) = r = p$), since both sides of the equation equal 0.

To derive the properties of the derivatives of the cutoff function $c(p,r,\tau,t)$, recall that the cutoff satisfies Equation (25). Under our assumptions, $G(c, p, r, \tau, t)$ is continuously differentiable in all its arguments, and as previously shown, $\frac{\partial G(c,p,r,\tau,t)}{\partial c} = 1 - M_v(c, r, \tau, t) > \epsilon > 0$. Hence, by the implicit function theorem,

$$c_p(p, r, \tau, t) = -\frac{\frac{\partial G(c,p,r,\tau,t)}{\partial p}}{\frac{\partial G(c,p,r,\tau,t)}{\partial c}} = -\frac{-1}{1 - M_v(c, r, \tau, t)} > 0$$

$$c_r(p, r, \tau, t) = -\frac{\frac{\partial G(c,p,r,\tau,t)}{\partial r}}{\frac{\partial G(c,p,r,\tau,t)}{\partial c}} = -\frac{-M_r(c, r, \tau, t)}{1 - M_v(c, r, \tau, t)} < 0$$

$$c_\tau(p, r, \tau, t) = -\frac{\frac{\partial G(c,p,r,\tau,t)}{\partial \tau}}{\frac{\partial G(c,p,r,\tau,t)}{\partial c}} = -\frac{-M_\tau(c, r, \tau, t)}{1 - M_v(c, r, \tau, t)} \leq 0.$$

Note that $M_r(c, r, \tau, t) < 0$, since increasing the reserve price strictly decreases the certainty equivalent of participating in the auction. Similarly, $M_\tau(c, r, \tau, t) \leq 0$, because increasing the length of the bidding phase τ increases the level of competition in the auction, hence decreasing the certainty equivalent. (The inequality is weak because $M_\tau(c, r, \tau, t) = 0$ when $c = r$.) Since $p \geq r$, the derivatives w.r.t. p and r should be interpreted as one-sided derivatives when $p = r$. Lastly, $c(p, r, \tau, t) = r$ when $p = r$ by our construction of $c(p, r, \tau, t)$. (When $p = r$, all bidders with $v > r$ prefer to accept the BP.) Moreover $c(p, r, \tau, t) > p$ when $p > r$, since we have just shown that $c_p(p, r, \tau, t) > 0$.

□

A.4 Proof of Proposition 4

See Online Appendix A.

A.5 Proof of Proposition 5

We have the differential equation

$$U''(c - r) = \frac{\Phi_r(c, r, \tau, t_1) + h(r, \tau, t_1)}{\Phi(c, r, \tau, t_1)} U'(c - r), \tag{26}$$

where

$$\Phi(c, r, \tau, t_1) = \alpha(r, \tau, t_1) \left[\frac{(1 - p_c(c, r, \tau, t_1))}{p_r(c, r, \tau, t_1)} - 1 \right].$$

We wish to prove that this differential equation has a unique solution $U(\cdot)$ under our assumptions, which include:

1. $U(\cdot)$ is twice continuously differentiable;

2. $U'(\cdot) > \epsilon$ for some $\epsilon > 0$;

3. $0 \geq U''(\cdot) > -C$, for some $0 < C < \infty$;

4. $U(0) = 0$, $U'(0) = 1$.

We can show that (13) has a unique solution even when r, τ, and t_1 are fixed. Hence, we assume $r = 0$, $\tau = \tau^*$, and $t_1 = t_1^*$. This results in a simple first-order linear ODE with a variable coefficient:

$$U''(c) = \Psi(c)U'(c), \tag{27}$$

where

$$\Psi(c) = \frac{\Phi_r(c, 0, \tau^*, t_1^*) + h(0, \tau^*, t_1^*)}{\Phi(c, 0, \tau^*, t_1^*)}.$$

has already been shown to be identified.

Since (27) implies that $\Psi(c) = \frac{U''(c)}{U'(c)}$, our assumed properties of U imply that the coefficient $\Psi(c)$ is integrable. It is well known that when $\Psi(c)$ is integrable, (27) has solutions in C^2 given by

$$\left|U'(c)\right| = k \exp\left(\int_0^c \Psi(c) dz\right), \tag{28}$$

where k is a constant that can be determined by the initial condition on $U'(\cdot)$. Since $U'(\cdot) > 0$ and we have the initial condition $U'(0) = 1$, it follows that the solution to (27) is unique and given by

$$U'(c) = \exp\left(\int_0^c \Psi(c) dz\right). \tag{29}$$

Since $U'(c)$ is identified, the initial condition $U(0) = 0$ identifies $U(\cdot)$. $\square$

A.6 Proof of Proposition 8

See Online Appendix A.

A.7 Proof of Proposition 9

See Online Appendix A.

A.8 Proof of Proposition 10

$\lambda(t_1)(1 - F_V(r))$ is identified over $r \in [\underline{r}, \overline{r}]$ and $t_1 \in [0, \overline{T})$ by the same arguemnts as in the previous two proofs.

Since W is observed and Z is observed (given $W = 1$), we can identify

$$\Pr\left(W = 1|p, r, \tau_0, t_1, B = 0\right),$$

$$\Pr\left(W = 1|z, p, r, \tau_0, t_1, B = 0\right),$$

and

$$p_Z(z|W = 1, p, r, \tau_0, t_1, B = 0),$$

over $r \in [\underline{r}, \overline{r}]$ and $p \in [p_0 - \epsilon, p_0 + \epsilon]$. The first two terms are the probability that the BP rejector wins the auction, with different conditions. The third term is the distribution of the final price given that the BP rejector wins the auction. (In general the conditional distribution of Z will have point mass at r, so we interpret conditional densities as being with respect to the sum of Lebesgue measure and counting measure at r.)

By Bayes' Theorem, we can write

$$p_Z(z|W = 1, p, r, \tau_0, t_1, B = 0) = \frac{p_y(z|p, r, \tau_0, t_1, B = 0)\Pr(W = 1|z, p, r, \tau_0, t_1, B = 0)}{\Pr(W = 1|p, r, \tau_0, t_1, B = 0)},$$

where p_y indicates the conditional density of $\tilde{Y}$. We can use this equation to recover $p_y(z|p, r, \tau_0, t_1, B = 0)$. This identifies $\alpha(r, \tau_0, t_1)$ and $h(y, \tau_0, t_1)$, since

$$\alpha(r, \tau_0, t_1) = \Pr(\tilde{Y} = r|p, r, \tau_0, t_1, B = 0),$$

and

$$h(y, \tau_0, t_1) = p_y(y|p, r, \tau_0, t_1, B = 0) \text{ for } r < y < c(p, r, \tau_0, t_1).$$

Since $\alpha(r, \tau_0, t_1)$ and $h(y, \tau_0, t_1)$ are identified, by the same arguments as in the previous proof, the Arrow-Pratt measure of risk aversion $\frac{U''}{U'}$ is identified at the point $c(p_0, r^*, \tau_0, t_1^*) - r^*$. By the same arguments as in the previous proof, $\delta(\cdot)$ is also identified at τ_0.

□

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